Abstract—Discrete-time system inversion for perfect tracking goes at the expense of intersample behavior. The aim of this paper is the development of a discrete-time inversion approach that improves continuous-time performance by also addressing the intersample behavior. The proposed approach balances the on-sample and intersample behavior and provides a whole range of new solutions, with stable inversion and multirate inversion as special cases. The approach is successfully applied to an LTI and an LPTV motion system. The proposed approach improves the intersample behavior through discrete-time system inversion.

Index Terms—Intersample behavior, discrete-time inversion, linear time invariant (LTI), linear periodically time varying (LPTV), stable inversion, multirate inversion

I. INTRODUCTION

Tracking control finds application in many areas, such as atomic force microscopes (AFMs) [1], wafer stages [2], and spectrometers [3]. The physical systems evolve in continuous time and hence their performance is naturally defined in continuous time. Many approaches for tracking control, including inverse model feedforward and iterative learning control (ILC), are based on system inversion. For continuous-time systems, system inversion approaches such as, for example, [4] can be used. However, controllers are often implemented in a digital environment since this provides a large flexibility at a low cost [5]. Due to the digital implementation, discrete-time control is often used.

One of the main challenges in system inversion is nonminimum-phase behavior. Causal inversion of nonminimum-phase systems yields unbounded inputs. To avoid unbounded inputs, many discrete-time inversion approaches have been proposed, see, e.g., [6] for a recent overview. Approximate inversion approaches such as ZPETC [7], ZMETC, and NPZ-Ignore [8] are well-known, but yield limited performance since an approximation is used. Optimal approaches such as norm-optimal feedforward, $\mathcal{H}_2$-preview control, and $\mathcal{H}_\infty$-preview control [6, Sections 4.3 and 4.4] yield high performance in discrete time. Discrete-time stable inversion [6, Section 4.2] yields exact tracking at the discrete-time samples.

Typically, discrete-time inversion approaches focus on the on-sample performance, i.e., at the discrete-time samples, resulting in poor intersample behavior, i.e., in between the samples, especially for zeros close to $z = -1$ [9]. This is observed for both linear time-invariant (LTI) and linear periodically time-varying (LPTV) systems [10]. As a consequence, the continuous-time behavior is poor. Indeed, the best on-sample performance does not necessarily lead to the best continuous-time performance.

Multirate inversion [11], [12], [13], [14] provides an interesting alternative to improve intersample behavior by sacrificing on-sample performance. However, the approach does not take into account the system dynamics when balancing the intersample and on-sample performance. As a consequence, the continuous-time performance will in general be suboptimal.

Although there exist many discrete-time inversion approaches, the balance between on-sample performance and intersample behavior is often not based on the system dynamics. The main contribution of this paper is a discrete-time inversion approach that finds the optimal balance between on-sample performance and intersample behavior for the purpose of continuous-time performance, for both LTI and LPTV systems. LPTV systems are of interest since they occur frequently, including sampled-data systems [5]; multirate systems [5], [15]; position-dependent systems with periodic tasks [16]; and non-equidistant sampling [17]. For both LTI and LPTV systems, the stable inversion and multirate inversion approaches are recovered as special cases. Related work includes [5], [18], [19] where synthesis-based approaches are presented. The approach presented in this paper does not require synthesis.

The outline of this paper is as follows. In Section II, the control diagram is presented and the control objective is formulated. The main idea of the proposed approach and preliminary results are presented in Section III. The proposed approach is presented in Section IV. The advantages of the approach are demonstrated by application to an LPTV motion system in Section V. Conclusions are presented in Section VI.

Notation. For notational convenience, single-input, single-output (SISO) systems are considered. The results can directly be generalized to square multivariable systems. Let $s^{(i)} \triangleq \frac{d^i s}{dt^i}$ denote the $i$th time-derivative of $s$, $\rho$ the Heaviside operator,
B(·) a bilinear transformation, and \( \mathbb{R}_{\geq a}^b = \{ x \in \mathbb{R}^b \mid x[k] > a \text{ for all } k = 0, 1, \ldots, b-1 \} \). Let \( \Sigma \hat{=} (A, B, C, D) \) be a state-space model and define \( T(\Sigma, T) \hat{=} (TAT^{-1}, TB, CT^{-1}, D) \).

II. Problem formulation

In this section, the control problem is formulated. The considered tracking control configuration is shown in Fig. 1, with reference trajectory \( r(t) \in \mathbb{R} \), control input \( u(t) \in \mathbb{R} \), output \( y(t) \in \mathbb{R} \), digital controller \( F \), sampler \( S \), and zero-order hold \( H \). The continuous-time, linear time-invariant (LTI) system \( H_c \) is given by

\[
\begin{align*}
\dot{x}(t) &= A_c x(t) + B_c u(t), \\
y(t) &= C_c x(t),
\end{align*}
\]

with \( x(t) \in \mathbb{R}^n, n \in \mathbb{N} \) and can be either an open-loop or closed-loop system.

In conventional discrete-time control, the focus is on on-sample performance. The discrete-time system \( H = SH_c H \) with \( H_c \) in (1) and sampling time \( \delta \) is given by

\[
\begin{align*}
x[k + 1] &= Ax[k] + Bu[k], \\
y[k] &= Cx[k],
\end{align*}
\]

with

\[
A = e^{A_c \delta}, \quad B = \int_0^\delta e^{A_c \tau} B_c d\tau, \quad C = C_c.
\]

In this setting, perfect on-sample tracking, i.e., \( e[k] = 0 \), for all \( k \), is achieved for \( F = H^{-1} \). However, this does not provide any guarantees for the intersample performance \( e(t), t \neq k\delta \). Hence, the continuous-time performance in terms of \( e(t) \), for all \( t \), may be poor as observed in, e.g., \([10]\).

In the next section, the main idea of the proposed approach and preliminary results are presented.

III. Conceptual idea and preliminary results

In this section, the conceptual idea of the approach and preliminary results are presented. The results form the basis for the complete approach presented in Section IV.

A. Conceptual idea

In the proposed approach, the system is decomposed into two parts and both parts are inverted separately according to Fig. 2, where \( H \) is decomposed as \( H = H_1 H_2 \). The inversion of system \( H_1 \) aims at the intersample behavior. More specific, let \( n_1 \) be the state dimension of \( H_1 \), then \( H_1 \) is inverted such that there is exact state tracking of a desired state \( \hat{x}_1[k] \) every \( n_1 \) samples. The inversion of \( H_2 \) aims at the on-sample behavior through perfect output tracking for every sample.

Exact state tracking is experienced to yield good intersample behavior in multirate inversion \([11]\), whereas exact output tracking yields good on-sample behavior in stable inversion \([6]\). Hence, the choice of the decomposition into \( H_1 \) and \( H_2 \) can be used to balance the on-sample behavior and the intersample behavior to the benefit of the continuous-time performance. The idea is conceptually illustrated in Fig. 3.

An important observation is that a small on-sample error does not necessarily yield a small continuous-time error. The figure shows that the proposed approach provides a whole range of solutions that were non-existing before. The stable inversion and multirate inversion solution are recovered as the two extreme cases, see also Section IV-C.

The proposed approach requires the decomposition \( H = H_1 H_2 \) in terms of state-space realizations and the desired state \( \hat{x}_1[k] \) for \( H_1 \), see also Fig. 2. In Section III-B, the desired state for the continuous-time system \( H_c \) is presented. In Section III-C, the state-space decomposition \( H = H_1 H_2 \) is presented. The results form the basis for the complete approach presented in Section IV.
B. Desired state for continuous-time system

In this section, the desired state for the continuous-time system is presented. Given a continuous-time reference trajectory \( r(t) \) together with its \( n-1 \) time derivatives and system \( H_c \) in (1), the objective is to determine a bounded state \( \hat{x}(t) \) such that \( y(t) = C_c \hat{x}(t) \) yields \( y^{(i)}(t) = r^{(i)}(t), \ i = 0, 1, \ldots, n-1 \), where \( (\cdot)^{(i)} \) denotes the \( i \)-th time derivative of \( (\cdot) \), i.e., such that \( \dot{r}(t) = \hat{y}(t) \) where

\[
\dot{r}(t) = \begin{bmatrix}
r^{(0)}(t) \\
r^{(1)}(t) \\
\vdots \\
r^{(n-1)}(t)
\end{bmatrix}, \quad \hat{y}(t) = \begin{bmatrix}
y^{(0)}(t) \\
y^{(1)}(t) \\
\vdots \\
y^{(n-1)}(t)
\end{bmatrix}.
\]  

(3)

A similar approach as in [20] is used based on the controllable canonical form given by Lemma 1, see also [21, Section 17.6]. The desired state is given by Theorem 2.

Lemma 1 (Controllable canonical form). Let the transfer function of \( H_c \) in (1) be given by

\[
H_c(s) = C_c(sI - A_c)^{-1}B_c = \frac{B(s)}{A(s)},
\]

(4a)

with

\[
A(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0,
\]

(4b)

\[
B(s) = b_ms^m + b_{m-1}s^{m-1} + \cdots + b_0,
\]

(4c)

\( b_0 \neq 0 \), then the controllable canonical form \( H_{ccf}(s) = \mathcal{T}(H_c, T_{ccf}) \) is given by

\[
\dot{x}_{ccf}(t) = A_{ccf}x_{ccf}(t) + B_{ccf}u(t),
\]

(5a)

\[
y(t) = C_{ccf}x_{ccf}(t),
\]

(5b)

where

\[
\begin{bmatrix}
A_{ccf} & B_{ccf} \\
C_{ccf} & 0
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
-a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} & b_0 \\
1 & s & s^2 & \cdots & s^{n-1} & 0
\end{bmatrix}
\]

(5c)

and

\[
T_{ccf}^{-1} = \begin{bmatrix}
B_c & A_cB_c & \cdots & A_c^{n-1}B_c
\end{bmatrix} = \begin{bmatrix}
a_0 & a_2 & \cdots & a_{n-1} \\
a_1 & a_3 & \cdots & 0 \\
a_2 & a_4 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix}.
\]

(6)

Theorem 2 (Desired continuous-time state). Let \( B^{-1}(s) \) in (4) be decomposed as

\[
B^{-1}(s) = F_s(s) + F_u(s),
\]

(7)

with all poles \( p_s \in \mathbb{C} \) of \( F_s(s) \) such that \( \Re(p_s) < 0 \) and all poles \( p_u \in \mathbb{C} \) of \( F_u(s) \) such that \( \Re(p_u) > 0 \). Let

\[
f_s(t) = L^{-1}(F_s(s)), \quad f_u(t) = L^{-1}(F_u(-s)),
\]

(8a)

\[
\hat{x}_{ccf,s}(t) = \int_{-\infty}^{t} f_s(t - \tau) \dot{r}(\tau) d\tau,
\]

(8b)

\[
\hat{x}_{ccf,u}(t) = \int_{t}^{\infty} f_u(t - \tau) \dot{r}(\tau) d\tau,
\]

(8c)

where \( L^{-1}(\cdot) \) is the inverse uni-lateral Laplace transform [22, Section 9.3]. Let \( H_c \) in (4) have realization (5), then \( y(t) = C_c \hat{x}(t) \) where

\[
\hat{x}(t) = T_{ccf}^{-1}(\hat{x}_{ccf,s}(t) + \hat{x}_{ccf,u}(t)),
\]

(9)

is bounded and such that \( \hat{y}(t) = \hat{r}(t) \), with \( \hat{y}(t), \hat{r}(t) \) in (3).

Proof: See [11, Section 3.2].

Theorem 2 provides the desired bounded state for optimal state tracking. Together with the state-space decomposition presented in the next section, Theorem 2 forms the basis of the proposed approach presented in Section IV.

Remark 1. If poles of \( B^{-1}(s) \) have \( \Re(p) = 0 \), i.e., \( B^{-1}(s) \) is non-hyperbolic, similar techniques as in [23] can be used.

C. State-space decomposition

In this section, the multiplicative state-space decomposition is presented. Together with Theorem 2, the decomposition forms the basis of the proposed approach in Section IV.

Given the state-space system \( H \) in (2), the interest is in minimal realizations \( H_1, H_2 \) such that \( H = H_1H_2 \) in terms of state-space realization, where the zeros and poles of \( H \) can be arbitrarily assigned to \( H_1 \) or \( H_2 \). The starting point is the multiplicative decomposition \( H = H_1H_2 \) in terms of transfer functions as given by Lemma 3.

Lemma 3 (Transfer function decomposition). Let \( H \equiv (A, B, C, D) \) be a state-space realization with \( n \) states and invertible \( D \). Let \( V \in \mathbb{R}^{n \times n_1} \) be a column space of an invariant subspace of \( A \) and let \( V_x \in \mathbb{R}^{n \times n_2} \) be a column space of an invariant subspace of \( A \) such that \( S = [V \ V_x] \) has full rank \( n = n_1 + n_2 \). Let

\[
\Pi = \begin{bmatrix}
I_{n_1} & 0_{n_1 \times n_2} \\
0_{n_2 \times n_1} & 0_{n_2 \times n_2}
\end{bmatrix}
\]

(10)

Then, the realizations

\[
H_{1f} \equiv \begin{bmatrix}
A & \Pi BD^{-1} \\
C & (I - \Pi)
\end{bmatrix}, \quad H_{2f} \equiv \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]

(11)

are such that \( H = H_{1f}H_{2f} \) in terms of transfer functions, i.e.,

\[
(C(zI - A)^{-1}B + D) = (C(zI - A)^{-1}\Pi BD^{-1} + I)(C(I - \Pi)(zI - A)^{-1}B + D).
\]

Proof: Follows directly from extending [24, Corollary 11] to \( D \neq I \).

If the \( D \) matrix in Theorem 4 is singular, a bilinear transformation [5, Section 3.4]; [22, Section 10.8.3] can possibly be employed to obtain an equivalent system with non-singular \( D \) matrix. A multiplicative decomposition for the transformed system is obtained through Lemma 3. Applying the inverse transformation on the decomposed system yields the decomposition for the original system since \( B(H_1H_2) = B(H_1)B(H_2) \).

Importantly, Lemma 3 guarantees equivalence in terms of transfer functions, but not in terms of state-space realizations. Indeed, the decomposition of Lemma 3 yields nonminimal realizations of \( H_{1f}, H_{2f} \) as both have state dimension \( n \). By exploiting the modal form and using a suitable state transformation, the desired state-space decomposition for the proposed approach is obtained as given by Theorem 4.
Theorem 4 (State-space decomposition). Let \( T_{mod} \in \mathbb{C}^{n \times n} \) be such that \( H_{mod} = T(H,T_{mod}) = (A, B, C, D) \) is in modal form [25, Section 7.4] with nonsingular \( D \). Let \( H_{1f}H_{2f} = H_{mod} \) be the decomposition given by Lemma 3. Let \( T_{per} \in \mathbb{R}^{n \times n} \) be such that

\[
T(H_{1f},T_{per}) \triangleq \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & 0 \\ C_1 & C_{1r} & I \end{bmatrix}, \quad (12)
\]

\[
T(H_{2f},T_{per}) \triangleq \begin{bmatrix} A_1 & 0 & B_{2r} \\ 0 & A_2 & 0 \\ C_2 & 0 & D \end{bmatrix}, \quad (13)
\]

with \( A_1 \in \mathbb{R}^{n_1 \times n_1}, A_2 \in \mathbb{R}^{n_2 \times n_2}, n_1 + n_2 = n \), and define

\[
H_1 \triangleq \begin{bmatrix} A_1 & B_1 \\ C_1 & I \end{bmatrix}, \quad H_2 \triangleq \begin{bmatrix} A_2 & B_2 \\ C_2 & D \end{bmatrix}. \quad (14)
\]

Furthermore, let \( X \in \mathbb{R}^{n_1 \times n_2} \) satisfy

\[
A_1X - XA_2 = B_1C_2. \quad (15)
\]

Then, the state-space realization of \( T(H_1,H_2,T_{per}^{-1},T_{12}) \) with

\[
T_{12} = \begin{bmatrix} I_{n_1} & X \\ 0_{n_1 \times n_2} & I_{n_2} \end{bmatrix}, \quad (16)
\]

is identical to that of \( H_{mod} \).

Proof: See Appendix A.

Theorem 4 yields a state-space decomposition \( H = H_1H_2 \) with identical state-space realizations. Note that such a decomposition always exists. Together with Theorem 2, Theorem 4 forms the basis for the proposed approach presented in the next section.

Remark 2. Note that \( V \) in Lemma 3 is related to the poles of \( H \), whereas \( V_x \) is related to the zeros of \( H \). Hence, \( V, V_x \) can be used to assign the poles and zeros to either \( H_1 \) or \( H_2 \).

Remark 3. The column spaces of the invariant subspaces in Lemma 3 can be constructed from eigenvectors. Note that for complex eigenvectors, the real and imaginary part should be used. For eigenvalues with multiplicity larger than 1, generalized eigenvectors obtained from the Jordan form can be used to ensure \( S \) has full rank.

Remark 4. Sylvester equation (15) has a unique solution \( X \) if the eigenvalues of \( A_1 \) and \( -A_2 \) are distinct [26].

IV. PROPOSED APPROACH

In the previous section, the global idea and preliminary results on the desired state and the state-space decomposition are presented. Based on these results, the proposed approach is presented. First, the approach for LTI systems is presented. Second, the approach for LPTV systems is presented. Finally, special cases are recovered.

A. Approach for LTI systems

The approach consists of two steps. First, stable inversion is applied to \( H_2 \) in (14) to obtain \( u[k] \) such that \( y_2[k] = u_1[k] \), for all \( k \), see also Fig. 2. The solution is given by Theorem 5 and provides exact output tracking every sample. See [6, Section 4.2] for a proof.

Theorem 5 (Inversion of \( H_2 \)). Consider Fig. 2 and let \( H_2^{-1} \) be given by

\[
\begin{align*}
x_s[k+1] &= A_s x_s[k] + B_s u_1[k], \quad (17a) \\
u[k] &= C_s x_s[k], \quad (17b)
\end{align*}
\]

with \( |\lambda(A_s)| < 1 \) and \( |\lambda(A_s)| > 1 \). Then, \( y_2[k] = u_1[k] \), for all \( k \), if

\[
u[k] = C_s x_s[k] \quad \text{and} \quad D u_1[k], \quad (18)
\]

which is bounded for bounded \( u_1 \) and where \( x_s \) follows from solving

\[
x_s[k+1] = A_s x_s[k] + B_s u_1[k], \quad x_s[-\infty] = 0 \quad (19)
\]

forward in time and \( x_u \) follows from solving

\[
x_u[k+1] = A_u x_u[k] + B_u u_1[k], \quad x_u[\infty] = 0 \quad (20)
\]

backward in time.

If \( u_1[k] \) is bounded, \( u[k] \) in Theorem 5 is bounded by construction of \( x_s[k], x_u[k] \), even if \( H_2 \) is nonminimum phase. The stable inversion solution in Theorem 5 achieves exact output tracking every sample and has infinite preactuation. Regular causal inversion is recovered as special case if the system is minimum phase (\( x_u \) is non-existing), see also [6].

Second, multirate inversion is applied to \( H_1 \) in (14) to obtain \( u_1[k] \). Note that by Theorem 5, \( y_2[k] = u_1[k] \), for all \( k \). The solution is based on lifting the state equation over \( n_1 \) samples. The solution is given by Theorem 6 and provides exact state tracking every \( n_1 \) samples.

Theorem 6 (Inversion of \( H_1 \)). Consider Fig. 2 with \( y_2[k] = u_1[k] \), for all \( k \), and let \( \hat{x}_1[k] \) be the desired state for system \( H_1 \) in (14). Consider the state equation lifted over \( \tau \) samples given by

\[
\hat{x}_1[q+1] = A_1 \hat{x}_1[q] + B_1 u_1[q], \quad (21)
\]

with \( A_1 = A_1^{n_1}, B_1 = [A_1^{n_1-1} B_1 \ A_1^{n_1-2} B_1 \ \ldots \ B_1]^T \),

\[
u_1[q] = u_1[kn_1] \quad u_1[kn_1+1] \quad \ldots \quad u_1[(k+1)n_1-1] \quad (22)
\]

and where \( x_1[q] = x_1[kn_1] \). Then, \( \hat{x}_1[q] = \hat{x}_1[q] \), for all \( q \), if

\[
u_1[q] = B_1^{-1}(\hat{x}_1[q+1] - A_1 \hat{x}_1[q]), \quad (22)
\]

which is bounded for bounded \( \hat{x}_1 \).

Proof: See [11, Section 3.3].

Importantly, the inversion approach in Theorem 6 is based on the continuous-time system \( H_c \) rather than the discrete-time system \( H \). The approach yields exact state tracking, and hence exact output tracking, every \( n_1 \) samples and has \( n_1 \) samples preactuation. Note that \( u_1 \) is bounded if \( \hat{x}_1 \), even if \( H_1 \) is nonminimum phase. More details can be found in, for example, [11], [13]. The desired state \( \hat{x}_1 \) in Theorem 6 is obtained by Procedure 7 which follows from Section III-B and Section III-C.
Procedure 7 (Desired state of $H_1$). Given $H_e$ in (5), $H$ in (2), and the decomposition $H = H_1H_2$ in Theorem 4, the following steps yield the desired state $\hat{x}_1[k]$ in Theorem 6.

1) Obtain the controllable canonical form $H_{ccf} = T(H_c,T_{eef})$ according to Lemma 1.
2) Obtain the desired state $\hat{x}(t)$ of $H_e$ using Theorem 2.
3) Set the desired state of $H$ to $\hat{x}[k] = \hat{x}(k\delta)$.
4) Obtain the desired state of $H_{mod}$: $\hat{x}_{mod}[k] = T_{mod}\hat{x}[k]$, with $H_{mod}$, $T_{mod}$ in Theorem 4.
5) Given $H_1$, $H_2$ in (14), let

$$H_{12} = H_1H_2 \begin{bmatrix} A_1 & B_1C_2 & B_1D_2 \\ C_1 & D_1C_2 & D_1D_2 \end{bmatrix}.$$

6) Obtain the desired state of $H_{12}$: $\hat{x}_{12}[k] = T_{12}^{-1}T_{per}\hat{x}_{mod}[k]$, with $T_{12}$ in (16) and $T_{per}$ satisfying (12) and (13).
7) Obtain the desired state for $H_1$: $\hat{x}_1[k] = [I_{n_1} \ 0 \ 0 \times n_2] \hat{x}_{12}[k]$.

The combination of the inversion of $H_2$ in Theorem 5 and the inversion of $H_1$ in Theorem 6 constitutes the control input $u[k]$ in Fig. 2, which is bounded by design, also for nonminimum-phase systems. The design freedom is in the decomposition of $H$ into $H_1$ and $H_2$ in Theorem 4. Equation (23) shows that the output is given by $y[k] = C_1x_1[k] + D_1x_2[k]$ since either $D_1 = 0$ or $D_2 = 0$ as $D = 0$ in (2). If $D_1 = 0$, $y[k] = C_1x_1[k]$ and since inversion of $H_1$ in Theorem 5 ensures perfect state tracking of $x_1[k]$ every $n_1$ samples, there is perfect output tracking every $n_1$ samples. If $D_1 \neq 0$, $y[k]$ also depends on $x_2[k]$ of $H_2$ and since inversion of $H_2$ in Theorem 6 does not provide perfect state tracking, there is no perfect output tracking for $y[k]$ every $n_1$ samples. Hence, to guarantee exact on-sample tracking every $n_1$ samples, $V, V_x$ in Theorem 4 are preferably chosen such that $D_1 = 0$.

In this section, the approach for LTI systems is presented. In the next section, the approach for LPTV systems is presented.

Remark 5. For strictly proper systems $H_2$, Theorem 5 can be applied to the bi-proper system $H_2$ obtained through time shifts $H_2 = \hat{s}^{d_2}H_2$, where $d_2$ is the relative degree of $H_2$, see also [6, Remark 1]. If there are eigenvalues on the unit circle, i.e., $\exists \lambda \in \lambda_1(A)$, then similar techniques as in [23] can be followed.

Remark 6. The decomposition of $H^{-1}$ given by (17) can be obtained through an eigenvalue decomposition.

Remark 7. Note that $B$ in Theorem 6 is the controllability matrix of $H_1$ and hence $B_{\perp}^{-1}$ exists if $H_1$ is controllable.

B. Approach for LPTV systems

In this section, the approach for LPTV systems is presented. Let the LPTV system $H$ with period $\tau \in \mathbb{N}$ be given by

$$x[k+1] = A[k]x[k] + B[k]u[k], \quad y[k] = C[k]x[k],$$

with $A[k+\tau] = A[k]$, $B[k+\tau] = B[k]$, $C[k+\tau] = C[k]$. LPTV systems may result from non-equidistant sampling as in Example 1.

Example 1 (Non-equidistant sampling). Let the sampling in Fig. 1 be non-equidistant in time and given by the sampling sequence $\Delta_{ne} \in \mathbb{R}_{>0}^\infty$ with periodicity $\tau \in \mathbb{N}$ defined as

$$\Delta_{ne} = (\delta_1, \delta_2, \ldots, \delta_\tau, \delta_1, \delta_2, \ldots),$$

with $\delta_i = \gamma_i\delta_0$, $\delta_0 \in \mathbb{R}_{>0}$, $\gamma_i \in \mathbb{N}$, $i = 1, 2, \ldots, \tau$. Then, the discretized system $H = SH\hat{H}$ is given by (24) with

$$A[i] = e^{A\delta_i}, \quad B[i] = \int_{0}^{\delta_i} e^{A\tau}B_c d\tau, \quad C = C_c,$$

where $i = 1, 2, \ldots, \tau$, where $A[k+\tau] = A[k]$, $B[k+\tau] = B[k]$, for all $k$. By linearity of $H_1$ and periodicity of $\Delta_{ne}$, $H$ is LPTV with period $\tau$.

The approach for LPTV systems is similar to that for LTI systems, with the key difference that an additional lifting step is used. The lifting step turns the LPTV system into a (multivariable) LTI system as given by Lemma 8.

Lemma 8. Lifting the input of $H$ in (24) over $\tau$ samples yields the LTI system $\hat{H}$ given by

$$\hat{x}[2] = A\hat{x} + Bu, \quad y = C\hat{x} + Du,$$

where

$$\hat{x} = [u[k\tau] \ u[k\tau+1] \ \ldots \ u[(k+1)\tau-1]],$$

with transition matrix

$$\Phi_{k+1,1} = \begin{bmatrix} \Phi_{\tau+1,2}B[1] & \Phi_{\tau+1,3}B[2] & \ldots & B[\tau] \\ C[1] & 0 & 0 & \ldots & 0 \\ C[2]\Phi_{2,1} & C[3]\Phi_{3,1}B[1] & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C[\tau]\Phi_{\tau,1} & C[\tau]\Phi_{\tau,2}B[1] & C[\tau]\Phi_{\tau,3}B[2] & \ldots & 0 \end{bmatrix},$$

with transition matrix

$$\Phi_{k_2,k_1} = \begin{cases} I, & k_2 = k_1, \\
A[k_2-1]A[k_2-2] \ldots A[k_1], & k_2 > k_1 \end{cases}$$

For the lifted system $\hat{H}$ in (27), the same approach as for the LTI system illustrated in Fig. 2 is used. The state-space decomposition $\hat{H} = \hat{H}_1\hat{H}_2$ is obtained using Theorem 4. System $\hat{H}_2$ is inverted using Theorem 5 and $\hat{H}_1$ is inverted using Theorem 6, where the desired state $\hat{x}_1[q]$ follows along the same lines as in Procedure 7. The result is the lifted input signal $\hat{u}[q]$, which, after inverse lifting, yields input $u[k]$, for the LPTV system $H$ in (24).

In the previous and present section, the approaches for LTI and LPTV are presented, respectively. Next, special cases are recovered.

C. Special cases

The proposed approach provides a whole range of solutions as illustrated in Fig. 3. The stable inversion and multirate inversion solution are recovered as the two extreme cases and given by Corollary 9 and Corollary 10. The results hold for both LTI and LPTV systems.
Corollary 9 (Special case: stable inversion [6], [27]). The stable inversion solution for $H$ is recovered from the proposed approach in Section IV as special case when $H = H_2$, i.e., $H_1 = I$ and $n_1 = 0$.

Corollary 10 (Special case: multirate inversion [11], [13]). The multirate inversion solution for $H$ is recovered from the proposed approach in Section IV as special case when $H = H_1$, i.e., $H_2 = I$ and $n_1 = n$.

Importantly, although Theorem 5 yields exact output tracking of $H_2$ for every sample, the inversion of $H_1$ does not reduce to conventional multirate inversion of $H$ since the desired state $\hat{x}_1[k]$ depends on the full system $H_c$ and not only on $H_1$.

Finally, the approach for LTI systems is recovered from that for LPTV systems as given by Corollary 11.

Corollary 11. The approach for LTI systems in Section IV-A is recovered as a special case from the approach for LPTV systems in Section IV-B for $\tau = 1$. Indeed, for $\tau = 1$, (2) is recovered from (24).

The proposed approach provides a whole range of solutions that were non-existing before, see also Fig. 3. The advantages are demonstrated by application to an LPTV motion system in Section V.

V. APPLICATION TO AN LPTV MOTION SYSTEM

In this section, the proposed approach is applied to an LPTV system resulting from non-equidistant sampling of an LTI motion system. The results show that many solutions of the proposed approach outperform the special cases of stable inversion and multirate inversion.

A. Setup

The considered motion system is the experimental high-precision positioning stage shown in Fig. 4(a). The Bode diagram of a frequency response function measurement is shown in Fig. 4(b). The identified 8th order continuous-time system $H_c$ ($n = 8$ and $m = 4$) is given by

$$H_c = \frac{4.576 \cdot 10^6}{s(s + 2.101)(s^2 + 10.89s + 3.665 \cdot 10^4)} \times \frac{(s^2 + 8.132s + 2.518 \cdot 10^4)(s^2 + 84.73s + 8.497 \cdot 10^5)}{(s^2 + 45.4s + 3.139 \cdot 10^5)(s^2 + 262.2s + 3.507 \cdot 10^9)}$$

and is stable and minimum phase. The Bode diagram of $H_c$ is also shown in Fig. 4(b).

The non-equidistant sampling sequence, see also Example 1, is set to $\gamma_1 = 1$, $\gamma_2 = 2$ ($\tau = 2$), with $\delta_0 = 400 \mu$s. The lifted system $\bar{H}$ in (27) has one nonminimum-phase (transmission) zero due to discretization. The trajectory $r(t)$ is given by the forward and backward motion shown in Fig. 5.

B. Simulation results

First, three different solutions are considered: the proposed approach with $n_1 = 4$, the special case $n_1 = 0$, i.e., multirate inversion in Corollary 10, and the special case $n_1 = 8$, i.e., stable inversion in Corollary 9. The results are shown in Fig. 6. The special case of stable inversion in Fig. 6(a) achieves perfect output tracking, however, the intersample performance is poor. The special case of multirate inversion in Fig. 6(b) yields perfect state tracking every $n = 8$ samples, with reasonable intersample performance. The proposed approach in Fig. 6(c) achieves perfect state tracking every $n_1 = 4$ samples and good intersample performance. The proposed approach outperforms the special cases of stable inversion and multirate inversion in terms of the continuous-time error $e(t)$.

Second, the performance is evaluated for a variety of solutions. The results are shown in Fig. 7 and quantify Fig. 3. The results show that many of the solutions provided by the proposed approach outperform the special cases of stable inversion and multirate inversion. Fig. 7 only shows results for
A discrete-time inversion approach is developed that allows to balance the on-sample and intersample behavior for the purpose of continuous-time performance. The approach is applicable to both LTI and LPTV systems. The multirate inversion and stable inversion approaches are recovered as special cases. Application to a motion system demonstrates the advantages of the proposed approach.

For LPTV systems, the proposed approach currently involves an additional lifting step, which limits applicability due to constraints on the input and state dimensions. In contrast, stable inversion and multirate inversion can be directly applied to LPTV systems. Future work focuses on an explicit state-space decomposition for LPTV systems to avoid the additional lifting step and thereby potentially increase the performance of the proposed approach.

APPENDIX
PROOF OF THEOREM 4
Due to the modal form of $H_{mod}$, the $A$ matrix of $H_{mod}$ is block diagonal and the states are decoupled per mode. The matrix $T_{per}$ is a permutation matrix and follows directly from $V,V_x$ and the state ordering of $H_{mod}$. The first $n_1$ states in (12) are uncontrollable and are redundant since the states are decoupled. Similarly, the last $n_2$ states in (13) are unobservable and are redundant since the states are decoupled. Hence, $H_1,H_2$ are minimal realizations such that $H_{mod} = H_1 H_2$ in terms of transfer functions. The product $H_1 H_2$ with $H_1,H_2$ in (14) is given by

$$H_1 H_2 = \begin{bmatrix} A_1 & B_1 C_2 & B_1 D \\ 0 & A_2 & B_2 \\ C_1 & C_2 & D \end{bmatrix}. \quad (30)$$

Using (15),

$$T(H_1 H_2, T_{12}) = \begin{bmatrix} A_1 & -A_1 X + B_1 C_2 + X A_2 & B_1 D + X B_2 \\ 0 & A_2 & B_2 \\ C_1 & C_1 + B_{2r} & D \end{bmatrix}. \quad (31a)$$

$$= \begin{bmatrix} A_1 & 0 & B_{2r} \\ 0 & A_2 & B_2 \\ C_1 & C_1 r & D \end{bmatrix}. \quad (31b)$$

By definition of $T_{per}$ in (12) and (13), $T(H_1 H_2, T_{per} T_{12}) = T(T(H_1 H_2, T_{12}), T_{per}^{-1}) = (A,B,C,D) = H_{mod}$ which concludes the proof.

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