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Numerically Reliable Identification of Fast Sampled Systems: A Novel δ-Domain Data-Dependent Orthonormal Polynomial Approach

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Abstract— **The practical utility of system identification algorithms is often limited by the reliability of their implementation in finite precision arithmetic. The aim of this paper is to develop a method for the numerically reliable identification of fast sampled systems. In this paper, a data-dependent orthonormal polynomial approach is developed for systems parametrized in the δ-domain. This effectively addresses both the numerical conditioning issues encountered in frequency-domain system identification and the inherent numerical round-off problems of fast-sampled systems in the common Z-domain description. Superiority of the proposed approach is shown in an example.**

I. Introduction

In parametric system identification, the central computational step typically involves solving a linear least squares problem, which is often ill-conditioned [1], [2], [3]. Several partial solutions to this conditioning problem have been proposed, including the use of frequency scaling [4], and the use of certain orthonormal bases such as Chebyshev polynomials, or Laguerre or Kautz filters [3], [5]. Another recent development is the use of data-dependent polynomial basis functions, which are orthonormal with respect to a data-dependent inner product [6], [7], [8]. These data-dependent orthonormal polynomials achieve optimal conditioning of the identification problem [8], i.e., condition number $\kappa = 1$.

Another development to improve numerical properties of various algorithms in identification and control involves replacing the traditional forward-shift operator, q, with the forward-difference operator, δ [9], [10]. The δ-operator transparently connects models in the discrete time domain and the continuous time domain [9]. Furthermore, using the δ-operator instead of the shift operator often leads to improved numerical performance when considering finite-word-length effects [10]. These advantages have been shown to be especially relevant for systems with fast sampling [11], i.e., where the sampling frequency is significantly higher than the dominant system dynamics, which is often the case in identification for control [12].

Although data-dependent orthonormal polynomials have been shown to provide significant numerical advantages in system identification, at present these advantages are restricted due to numerical issues, such as finite-word-length effects, in other essential computation steps. The aim of this paper is to combine the numerical advantages of the discrete-time δ-domain description and data-dependent orthonormal polynomials to obtain a numerically reliable discrete-time system identification approach for fast sampled systems. The main contributions of this paper are the following.

1) An approach for numerically reliable identification in the δ-domain, using newly developed theory for the efficient construction of δ-domain orthonormal polynomials with respect to a data-dependent inner product.
2) An example revealing the superior numerical performance of the δ-domain approach compared to the existing Z-domain approach for fast-sampled systems.

The algorithms developed in the literature for the efficient construction of polynomials that are orthonormal with respect to a data-dependent inner product, are typically limited either to the case of discrete inner products with the nodes of the discrete inner product on the real line or on the imaginary axis [13], or with nodes on the unit circle [14]. For the δ-domain case considered in this paper, the nodes lie on circles in the complex plane for which these existing efficient solutions do not apply. In this paper, the efficient construction of orthonormal polynomials from spectral data on these circles in the complex plane is considered. Thereby enabling the numerically reliable identification of fast-sampled systems in the δ-domain.

II. Problem formulation

The problem considered in this paper is that of numerically reliable frequency-domain identification of linear systems with fast sampling. First, the problem of frequency domain system identification is defined. Second, the numerical challenges for fast-sampled systems are highlighted.

A. Frequency-domain system identification

The goal for system identification is to estimate an appropriate system model, $\hat{G}(\xi)$, describing the relevant behavior of a physical process. In frequency domain identification this is done using frequency domain input-output data of the system, typically obtained using a digital measurement environment. In this paper, identification of single-input single-output (SISO) systems is considered. The extension to multiple-input multiple-output (MIMO) systems follows along similar lines as in, e.g., [15].

In this paper, linear time invariant (LTI) systems are considered, represented by real-rational transfer functions, i.e., $\hat{G}(\xi) \in \mathscr{R}$. Here, ξ, is an indeterminate frequency variable which depends on the identification domain. Traditionally for the continuous-time Laplace domain $\xi = s = j\omega$, and

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for the discrete-time Z-domain $\xi = z = e^{j\omega T_s}$, where ω is the frequency variable and T_s is the sampling interval. The model $\hat{G}(\xi)$ is parametrized as

$$\hat{G}(\xi,\theta) = \frac{\hat{N}(\xi,\theta)}{\hat{D}(\xi,\theta)}, \tag{1}$$

where $\hat{N}(\xi,\theta), \hat{D}(\xi,\theta) \in \mathbb{R}[\xi]$, which are linearly parametrized with respect to a set of polynomial basis functions $\{\phi_j(\xi)\}_{j=0}^{n}$, i.e.,

$$\begin{bmatrix}\hat{D}(\xi,\theta)\\ \hat{N}(\xi,\theta)\end{bmatrix} = \begin{bmatrix}\phi_0(\xi) & \phi_1(\xi) & \cdots & \phi_n(\xi)\end{bmatrix}\theta, \tag{2}$$

with $\theta \in \mathbb{R}^{n\times 1}$ and $\phi_j(\xi) \in \mathbb{R}^{2\times 1}[\xi]$.

The optimal model within this parametrization is then selected by minimizing a suitable error criterion, e.g., $\hat{G}_{\text{opt}}(\xi) = \hat{G}(\xi,\theta_{\text{opt}})$, where

$$\theta_{\text{opt}} = \arg\min_{\theta} \sum_{i=1}^{m} \left|\tilde{w}_i \left(\tilde{G}(\xi_i) - \hat{G}(\xi_i,\theta)\right)\right|^2, \tag{3}$$

where $\tilde{w}_1,\ldots,\tilde{w}_m$ is a sequence of weights and where $\tilde{G}(\xi_i)$ is the identified Frequency Response Function (FRF) of the system for the i-th frequency point. Solving (3) involves solving a nonlinear optimization problem due to the rational parametrization of $\hat{G}(\xi)$. Proposed solutions include solving a linearized version of (3) that is obtained by multiplying the problem with the denominator polynomial $\hat{D}(\xi,\theta)$, known as Levy's method [1], or the use of algorithms such as the Sanathanan-Koerner (SK) algorithm [2] or Gauss-Newton algorithm [16] to iteratively reformulate and solve the nonlinear problem (3). For additional details, see, e.g., [15].

In these approaches and algorithms, the problem being solved is a weighted polynomial least squares problem of the form,

$$\hat{\theta} = \arg\min_{\theta} \|\boldsymbol{w} f(\boldsymbol{\xi},\theta)\|_2^2, \tag{4}$$

with weight and node vectors,

$$\boldsymbol{w} = [w_1^T, w_2^T, \ldots, w_m^T]^T, \quad \boldsymbol{\xi} = [\xi_1, \xi_2, \ldots, \xi_m]^T, \tag{5}$$

and where

$$\|\boldsymbol{w} f(\boldsymbol{\xi},\theta)\|_2^2 := \sum_{i=1}^{m} f(\xi_i,\theta)^H w_i^H w_i f(\xi_i,\theta), \tag{6}$$

with $w_i \in \mathbb{C}^{\alpha\times 1}$, and

$$f(\xi_i,\theta) = p(\xi_i,\theta) - y_i, \tag{7}$$

where

$$p(\xi,\theta) = \begin{bmatrix}\phi_0(\xi) & \phi_1(\xi) & \cdots & \phi_n(\xi)\end{bmatrix}\theta, \tag{8}$$

and where $y_i \in \mathbb{C}^{\alpha\times 1}$ is determined by the problem data and parameter constraints. The problem that is considered in this paper is of this form, where for clarity only the scalar polynomial case, i.e., $p(\xi,\theta) \in \mathbb{R}[\xi]$ and $w_i \in \mathbb{C}$, is considered in the remainder of this paper. Extension to the general vector polynomial case, i.e., $p(\xi,\theta) \in \mathbb{R}^{\alpha\times 1}[\xi]$ follows along similar lines as is explained in Section V.

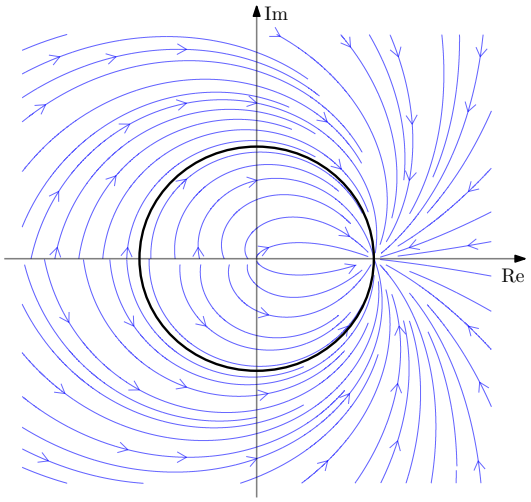

Fig. 1. Streamlines for the vector field $\mathrm{d}z/\mathrm{d}f_s \propto -z\ln(z)$ showing the attraction of point $z = 1$ as the sampling frequency increases.

B. Numerical challenges for fast-sampled systems

In the identification problem as stated in Section II-A, ξ in (1) still remains to be specified. Typically, the Laplace domain, $\xi = s$, is used when the system is in continuous time, whereas the Z-domain, $\xi = z$, is used when the system is in discrete time. For fast-sampled systems, i.e., when the sampling frequency of the digital measurement environment is fast relative to the dominant dynamics of the underlying physical process, the discrete Z-domain description is known to suffer from numerical issues.

To analyze the numerical issues that arise for fast-sampled systems in the Z-domain, the derivative of the complex variable $z = e^{s/f_s}$ with respect to the sampling frequency $f_s = 1/T_s$ is considered,

$$\frac{\mathrm{d}z}{\mathrm{d}f_s} = \frac{\mathrm{d}(e^{s/f_s})}{\mathrm{d}f_s} = -\frac{e^{s/f_s}s}{f_s^2} = -\frac{z\ln(z)}{f_s} \propto -z\ln(z), \tag{9}$$

where $s = f_s\ln(z)$ is used in the third step and where $\propto$ denotes proportionality. Streamlines for the vector field $-z\ln(z)$ are plotted in the complex plane in Figure 1. This figure clearly shows that all streamlines converge to the point $z = 1$. For fast-sampled systems this essentially means that the poles describing the system dynamics will become concentrated near the point $z = 1$.

This concentration around $z = 1$ leads to a numerical loss of significance due to round-off effects in finite precision arithmetic as a large part of the available finite wordlength is used to store this unity part of the relevant parameters, which contains no relevant information. These numerical problems can be overcome by considering the discrete δ-domain instead of the Z-domain, which is proposed in this paper as an essential steps towards obtaining a numerically reliable identification procedure for fast-sampled system.

III. Numerically reliable identification for fast sampled system

In this section, it is first shown how the discrete δ-domain description leads to a reduction of the numerical round-off issues as seen in Section II-B. Next, the problem of numerically reliable identification using data-dependent orthonormal polynomials is considered and it is shown how this framework can be extended to include the δ-domain case. Last, the proposed identification approach for numerically reliable identification in the δ-domain is presented.

A. The discrete δ-domain: reducing round-off errors

As an alternative to the Z-domain description, the discrete-time system can be represented in the δ-domain, where the δ-operator

$$\delta = \frac{q-1}{T_s}, \tag{10}$$

is used instead of the shift operator q, see [9]. The main numerical advantage of this δ-domain description is that it shifts the point $z=1$ to the origin, solving the associated loss of significance problems. Additionally, the δ operator intuitively connects the discrete and continuous time domains as for a differentiable function $x(t)$

$$\lim_{T_s\to 0} \delta x(t) = \lim_{T_s\to 0} \frac{x(t+T_s)-x(t)}{T_s} = \frac{\mathrm{d}x}{\mathrm{d}t}, \tag{11}$$

meaning that as the sampling frequency increases, the δ-domain description converges to the continuous time Laplace domain description. This provides insight and gives a certain confidence that even for sampling time approaching zero the discrete-time description in the δ-domain remains well behaved. The δ-domain representation often improves numerical aspects especially for fast sampled systems, e.g., as in the control approaches in [9], [10], [11].

B. Numerically reliable identification using data-dependent orthonormal polynomials

Solving the polynomial least-squares problem, (4), is equivalent to determining the least-squares solution to

$$W\Phi_n\theta = W\boldsymbol{y}, \tag{12}$$

with

$$W = \mathrm{diag}(w_1, w_2, \ldots, w_m), \tag{13}$$

$$\Phi_n = \begin{bmatrix} \phi_0(\xi_1) & \phi_1(\xi_1) & \cdots & \phi_n(\xi_1) \\ \phi_0(\xi_2) & \phi_1(\xi_2) & \cdots & \phi_n(\xi_2) \\ \vdots & \vdots & & \vdots \\ \phi_0(\xi_m) & \phi_1(\xi_m) & \cdots & \phi_n(\xi_m) \end{bmatrix}, \tag{14}$$

$$\boldsymbol{y} = \begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix}^T. \tag{15}$$

The key observation is that the matrix $W\Phi_n$ in (12) can be severely ill-conditioned depending on the choice of basis functions, thereby deteriorating the performance of the identification algorithms [8].

Several approaches have been proposed in literature to mitigate this conditioning problem [3], [4], [5], confirming that this is an important aspect in frequency domain system identification. The approaches in [3], [5] focus on a change of basis functions to obtain a Φ_n-matrix with advantageous numerical properties. However, this does not guarantee an improvement in the conditioning of $W\Phi_n$, which is the relevant problem matrix for solving (12). To guarantee that $W\Phi_n$ is well-conditioned, the problem data, as contained in W, needs to be taken into account in the choice of basis functions, i.e., the basis functions should be data-dependent.

A key result in, e.g., [7], shows that optimal conditioning of $W\Phi_n$ can indeed be achieved using a set of polynomials, $\{\phi_j(\xi)\}_{j=0}^n$, which are orthonormal with respect to the following discrete inner product

$$\langle \phi_p, \phi_l \rangle = \sum_{i=1}^{m} (w_i \phi_p(\xi_i))^* w_i \phi_l(\xi_i), \tag{16}$$

where x^* denotes the complex conjugate of x. The orthonormality condition of $\{\phi_j(\xi)\}_{j=0}^n$ with respect to (16) yields

$$\Phi_n^H W^H W \Phi_n = I_n, \tag{17}$$

meaning $Q = W\Phi_n$ is a semi-unitary matrix and therefore the least squares problem (12) can be solved using

$$\hat{\theta} = (W\Phi_n)^H W\mathbf{y}, \tag{18}$$

with $\kappa(W\Phi_n) = 1$.

The orthonormal polynomials, $\{\phi_j(\xi)\}_{j=0}^n$, can be constructed from the general recurrence relation

$$h_{k,k-1}\phi_k(\xi) = \xi\phi_{k-1}(\xi) - \sum_{j=0}^{k-1} \phi_j(\xi) h_{j,k-1}, \tag{19}$$

where the recurrence parameters $h_{i,j}$ can be considered as the elements of an upper Hessenberg matrix H. This Hessenberg recurrence matrix can be obtained from the problem data by solving the following inverse eigenvalue problem.

Definition 1 (Inverse eigenvalue problem): Given the set of points ξ_i and weights w_i, $i = 1, 2, \ldots, m$, compute an upper Hessenberg matrix H, with subdiagonal elements $h_{k+1,k} \in \mathbb{R}_{>0}$, such that

- it is unitarily similar to the diagonal matrix $X = \mathrm{diag}(\xi_1, \xi_2, \ldots, \xi_m)$, i.e.,

$$Q^H X Q = H, \tag{20}$$

- the first column of Q equals $\dfrac{\boldsymbol{w}}{\|\boldsymbol{w}\|_2}$, with $\boldsymbol{w}$ as in (5).

For details on the relation between orthonormal polynomials, (discrete) inner-products, Hessenberg matrices, inverse eigenvalue problems and their solution algorithms see, e.g., [17, Chapter 12], [13], [14].

Solving the inverse eigenvalue problem of Definition 1, generally requires $\mathcal{O}(mn^2)$ operations. However, for problems in the s-domain, where all points ξ_i lie on the imaginary axis, and the Z-domain, where all points ξ_i lie on the unit circle, this computational complexity can be reduced by an order of n. This reduction is the result of the additional structural properties of the Hessenberg matrix H for these

particular cases. In the case where all points ξ_i lie on the imaginary axis, H is tridiagonal leading to a reduction in the computation complexity of operations involving this tridiagonal matrix, see, e.g., [13]. In the case where all points ξ_i lie on the unit circle, H is unitary which also leads to a reduction in the computational complexity as a result of the Schur parametrization for unitary Hessenberg matrices, see, e.g., [14]. Furthermore, in both these cases, simplified recurrence relations, as compared to (19), exist which also enable the recursive construction of the polynomial basis functions to be performed in $\mathcal{O}(mn)$ operations.

The existence of additional structural properties of the Hessenberg recurrence matrix H that allow such an order of n reduction in computational complexity for both the Laplace-domain and Z-domain cases, gives rise to the question whether similar structural properties exist for the δ-domain case. This hypothesis is strengthened by the fact that the δ-domain can be seen as a natural connection between the continuous-time and discrete-time cases, as explained in Section III-A. In the following section, it is shown that such structural properties indeed exist and how these too lead to an efficient construction algorithm for data-dependent orthonormal polynomials in the δ-domain.

C. Orthonormal polynomials in the δ-domain: a quasi-separable matrix approach

In the δ-domain, all points ξ_i are given by,

$$\xi_{\delta,i} = \frac{e^{j\omega_i T_s} - 1}{T_s}. \tag{21}$$

The question answered in this section is whether there are additional structural properties for the Hessenberg recurrence matrix H when all points ξ_i are defined as in (21), such that the following requirements are met.

- R.1 H can be parametrized using $\mathcal{O}(n)$ parameters.
- R.2 There exist recurrence relations that enable the computation of orthonormal polynomials in $\mathcal{O}(mn)$ operations.
- R.3 The inverse eigenvalue problem of Definition 1 can be solved in $\mathcal{O}(mn)$ operations.

To find a matrix structure satisfying these requirements, a class of structured matrices is considered that generalizes the matrix structures encountered in the s-domain and Z-domain cases. This is the class of $(H,1)$-quasiseparable matrices, which indeed encompasses both the tridiagonal matrices encountered in the s-domain case as well as the unitary Hessenberg matrices encountered in the Z-domain case, as is shown in [18]. This class of matrices is defined as follows.

Definition 2 ($(H,1)$-quasiseparable matrices): An upper Hessenberg matrix H is $(H,1)$-quasiseparable if,

$$\max_{1 \le i \le n-1} \operatorname{rank}\left(H(1:i, i+1:n)\right) = 1. \tag{22}$$

Here, the notation $A(i_1:i_2, j_1:j_2)$ is used to denote the submatrix of A containing rows i_1 to i_2 and columns j_1 to j_2.

This general class of rank-structured Hessenberg matrices also includes the δ-domain case as shown in the following new result. Consider (20) in the Z-domain case,

$$Q^H X_z Q = H_z, \tag{23}$$

for the δ-domain it follows from (21) that

$$H_\delta = Q^H X_\delta Q = \frac{1}{T_s} Q^H (X_z - I) Q = \frac{1}{T_s}(H_z - I), \tag{24}$$

which has the same off-diagonal structure as H_z. Therefore since H_z is $(H,1)$-quasiseparable matrices it follows from (24) that the Hessenberg recurrence matrix, H_δ, for the δ-domain case is also $(H,1)$-quasiseparable.

For the class of $(H,1)$-quasiseparable matrices, there exists a generator description using $\mathcal{O}(n)$ parameters, as is shown in the following theorem.

Theorem 1 ($(H,1)$-quasiseparable generators): An upper Hessenberg matrix is $(H,1)$-quasiseparable matrix if and only if it can be described as

$$H = \begin{bmatrix} d_1 & g_1h_2 & g_1b_2h_3 & \cdots & \cdots & g_1b_{(2:n-1)}h_n \\ a_1 & d_2 & g_2h_3 & \cdots & \cdots & g_2b_{(3:n-1)}h_n \\ 0 & a_2 & d_3 & \cdots & \cdots & g_3b_{(4:n-1)}h_n \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-2} & d_{n-1} & g_{n-1}h_n \\ 0 & \cdots & \cdots & 0 & a_{n-1} & d_n \end{bmatrix}, \tag{25}$$

where $b_{(x:y)} = \prod_{i=x}^{y} b_i$ and where $a_i, b_i, d_i, g_i, h_i \in \mathbb{C}$ are called the generators of H.

For a proof see [19, Theorem 3.5]. This parametrization fulfills the requirement R.1.

Using this generator description, simplified recurrence relations can be formulated for the polynomial system related to $(H,1)$-quasiseparable matrices through (19), as is shown in, e.g., [18]. This recurrence relation is given by

$$\begin{bmatrix} F_0(\xi) \\ \phi_0(\xi) \end{bmatrix} = \frac{1}{\|\boldsymbol{w}\|_2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{26}$$

$$\begin{bmatrix} F_k(\xi) \\ a_k\phi_k(\xi) \end{bmatrix} = \begin{bmatrix} b_k & -g_k \\ h_k & \xi - d_k \end{bmatrix} \begin{bmatrix} F_{k-1}(\xi) \\ \phi_{k-1}(\xi) \end{bmatrix}. \tag{27}$$

where $F_k(\xi)$ is a set of auxiliary polynomials. This recurrence relation fulfills the requirement R.2.

Solving the inverse eigenvalue problem of Definition 1 is often done using the chasing down the diagonal approach [13], where m update steps are performed and where each update step involves chasing a single bulge-element down the diagonal of the Hessenberg matrix. This chasing is performed using $\mathcal{O}(n)$ Givens rotations and the application of a single Givens rotation generally has a computational complexity of $\mathcal{O}(n)$. For the Laplace-domain and Z-domain cases, the matrix structure allows a single Givens operation to be performed with a computational complexity $\mathcal{O}(1)$. This reduction in computational complexity is also possible for the δ-domain due to the quasi-separable structure of the Hessenberg recurrence matrix. This results from the fact that applying the Givens operation to the rank one blocks from upper diagonal part of the $(H,1)$-quasiseparable recurrence

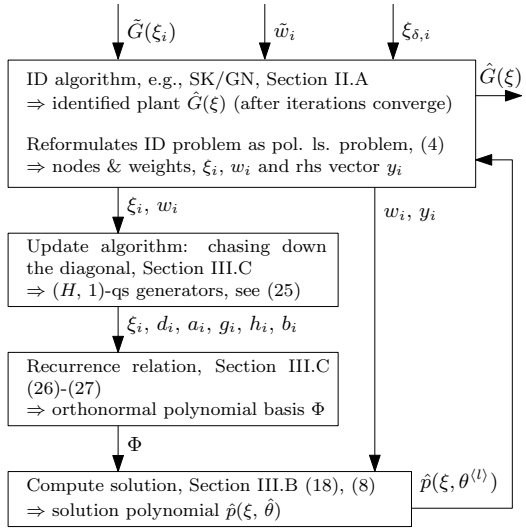

Fig. 2. Unified numerically reliable identification approach.

matrix can be done using $\mathcal{O}(1)$ operations. Furthermore, as a result of the rank symmetry of unitary matrices, see [20, Theorem 5], the quasiseparable structure is maintained during chasing. This shows that requirements R.1–R.3 are fulfilled, enabling the efficient construction of data-dependent orthonormal polynomials for δ-domain identification.

D. Overall identification approach

The δ-domain identification approach as proposed in this paper is summarized in Figure 2. In particular, it is shown how, starting with FRF data of a given system, a system model is identified using a class of well known identification algorithms in combination with numerically optimal polynomial basis functions expressed in the δ domain. The identification approach is implemented in MATLAB. Additional implementation details for the proposed approach are omitted for brevity and are to be published elsewhere.

IV. Example

In this section the numerical advantages of the proposed approach are shown using an example based on a real system.

A. System description

The simulation example used here is based on a model of an Active Vibration Isolation System (AVIS) as detailed in [21]. In this model a rigid isolated payload is connected to a rigid base frame with four isolator modules, each modeled as three orthogonal linear springs and dampers. The isolated payload has six rigid-body degrees of freedom which leads to a model order of 12. In this paper the SISO transfer function is considered that represents the coupling between a moment, M_x, applied in the θ_x direction and the resulting rotation in the θ_y direction.

B. Methods

The true system $G_0(\xi)$ is given by,

$$G_0(\xi) = \frac{n_0(\xi)}{d_0(\xi)}. \tag{28}$$

In this simulation example, the denominator polynomial, $d_0(\xi)$, is estimated from simulated FRF data of the system, $\tilde{G}(\xi_i)$, and it is assumed that the true numerator polynomial, $n_0(\xi)$, is known. This is done to reduce the identification problem to a scalar polynomial least-squares problem.

The simulated FRF data $\tilde{G}(\xi_i)$ is given as,

$$\tilde{G}(\xi_i) = G_0(\xi_i) + N_i \tag{29}$$

where N_i is randomly generated, circularly complex normally distributed noise with zero mean and a constant variance with a value of 0.1 times the median absolute value of $G_0(\xi_i)$. In this example 10^3 frequency points, ω_i, are considered which are logarithmically spaced between 0.5 and 10 Hz. This band-limited example is used to clearly demonstrate the numerical problems that occur in the discrete Z-domain for fast sampled systems, where only a small part of the complex plane around $z = 1$ is relevant in the identification problem. This concentration around $z = 1$ is also expected to occur in broadband complex systems containing significant low-frequency dynamics, as for these systems the weights w_i in (16) tend to be concentrated in the low-frequency region.

To identify the denominator polynomial from this data, ten iterations of the SK-algorithm are used where $\hat{N}(\xi_i, \theta)$ is replaced with $n_0(\xi_i)$. This identification procedure is performed using two different approaches. First, in the Z-domain using the Schur parametrization, implemented using a rotation-chasing algorithm similar to [22]. Second, in the δ-domain, using the proposed approach of Section III-D.

The identification approaches are implemented and performed for increasing sampling frequencies in single precision floating-point arithmetic. To compare the numerical performance of the approaches, the conditioning of the matrix $Q = W\Phi$ is considered, which should be equal to 1 when the orthonormalization procedure is successful. To quantify the orthonormalization performance, the geometric mean of $\kappa(Q) - 1$ over all iterations is considered, i.e.,

$$\mu_g\{\kappa(Q) - 1\} = \left(\prod_{l=1}^{10} \left(\kappa(Q^{\langle l \rangle}) - 1 \right) \right)^{1/10}. \tag{30}$$

Whenever an invalid number, i.e., $\pm\infty$ or NaN, is encountered it is replaced by the maximum conditioning number for the purpose of determining $\mu_g\{\kappa(Q) - 1\}$.

C. Results

In Figure 3, the conditioning results, i.e., $\mu_g\{\kappa(Q) - 1\}$, are shown for different sampling frequencies. The results in this figure can be interpreted as a measure for the numerical loss of significance encountered in the orthonormaliztion procedure. From this figure it is clear that the numerical performance of the δ-domain approach is superior to that of the Z-domain approach. Furthermore it is clear that for the δ-domain approach, this loss of significance does not scale with the sampling frequency, whereas for the Z-domain approach it does. In fact, for the Z-domain approach the numerical loss of accuracy initially scales quadratically with the sampling frequency. After the initial quadratic scaling,

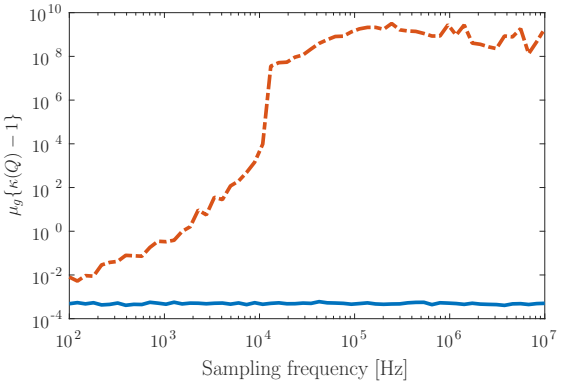

Fig. 3. Geometric mean of the numerical conditioning error $\kappa(Q)-1$ for different sampling frequencies, in single precision arithmetic. Shows the results for the Z-domain approach with Schur parametrization (red dash-dotted), which quickly deteriorate for increasing sampling frequencies, and the δ-domain approach proposed in this paper (blue solid), which consistently yields superior results.

a sharp increase in the numerical loss of accuracy can be observed once $\kappa(Q)-1>1$, which can be viewed as a threshold after which a loss of orthogonality occurs. After this sharp increase a plateau is reached for $\kappa(Q)$ which is in the order of $1/\varepsilon$, where ε is equal to the machine precision, this indicates a total loss of numerical significance.

The difference in the numerical performance of the considered approaches can be explained from the fact that, as the sampling frequency increases, the discrete inner product (16) becomes concentrated around $z=1$ for the Z-domain case. This causes the Hessenberg recurrence matrix $H_z=Q^H X_z Q$ to converge to the identity matrix, leading to the numerical issues as discussed in Section II-B.

V. Conclusions and Outlook

This paper enables the numerically reliable identification of fast-sampled systems by utilizing a δ-domain description and solving the least-squares estimation problem using newly developed data-dependent orthonormal polynomials. In Section III-C, it is shown how earlier results for the s-domain and Z-domain can be extended to obtain an approach for the fast construction of data-dependent orthonormal polynomials in the δ-domain using quasiseparable Hessenberg matrices.

In Section IV, an example based on a Active Vibration Isolation System is presented. From the results of this example it can be concluded that the numerical performance of the proposed δ-domain approach is superior to that of the existing Z-domain approach.

Throughout this paper, the scalar polynomial least-squares problem has been considered. However, to identify rational system models and to be able to handle systems with multiple inputs and outputs, the problems that need to be solved involve vector polynomials, as is shown in, e.g., [7]. To extend the results shown in this paper to the vector polynomial case involves extending the scalar quasiseparable generator description to a block-generator description as in, e.g., [23, Chapter 12] and adjusting the chasing operations to accommodate this block quasiseparable structure.

References

[1] E. C. Levy, "Complex-curve fitting," *IRE Transactions on Automatic Control*, vol. AC-4, no. 1, pp. 37–43, 1959.

[2] C. K. Sanathanan and J. Koerner, "Transfer function synthesis as a ratio of two complex polynomials," *IEEE Transactions on Automatic Control*, vol. 8, no. 1, pp. 56–58, 1963.

[3] B. Ninness and H. Hjalmarsson, "Model structure and numerical properties of normal equations," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 48, no. 4, pp. 425–437, apr 2001.

[4] R. Pintelon and I. Kollár, "On the frequency scaling in continuous-time modeling," *IEEE Transactions on Instrumentation and Measurement*, vol. 54, no. 1, pp. 318–321, 2005.

[5] B. Wahlberg and P. Mäkilä, "On approximation of stable linear dynamical systems using Laguerre and Kautz functions," *Automatica*, vol. 32, no. 5, pp. 693 – 708, 1996.

[6] Y. Rolain, R. Pintelon, K. Xu, and H. Vold, "Best conditioned parametric identification of transfer function models in the frequency domain," *IEEE Transactions on Automatic Control*, vol. 40, no. 11, pp. 1954–1960, 1995.

[7] A. Bultheel, M. V. Barel, Y. Rolain, and R. Pintelon, "Numerically robust transfer function modeling from noisy frequency domain data," *IEEE Transactions on Automatic Control*, vol. 50, no. 11, pp. 1835–1839, 2005.

[8] R. van Herpen, T. Oomen, and M. Steinbuch, "Optimally conditioned instrumental variable approach for frequency-domain system identification," *Automatica*, vol. 50, no. 9, pp. 2281 – 2293, 2014.

[9] G. Goodwin, R. Leal, D. Mayne, and R. Middleton, "Rapprochement between continuous and discrete model reference adaptive control," *Automatica*, vol. 22, no. 2, pp. 199 – 207, 1986.

[10] R. Middleton and G. Goodwin, "Improved finite word length characteristics in digital control using delta operators," *IEEE Transactions on Automatic Control*, vol. 31, no. 11, pp. 1015–1021, November 1986.

[11] G. Li and M. Gevers, "Comparative study of finite wordlength effects in shift and delta operator parameterizations," *IEEE Transactions on Automatic Control*, vol. 38, no. 5, pp. 803–807, May 1993.

[12] M. Gevers and G. Li, *Parametrizations in the Delta operator*. London: Springer London, 1993, pp. 289–318.

[13] W. B. Gragg and W. J. Harrod, "The numerically stable reconstruction of Jacobi matrices from spectral data," *Numerische Mathematik*, vol. 44, no. 3, pp. 317–335, 1984.

[14] G. Ammar, W. Gragg, and L. Reichel, "Constructing a unitary Hessenberg matrix from spectral data," in *Numerical Linear Algebra, Digital Signal Processing and Parallel Algorithms*, G. H. Golub and P. Van Dooren, Eds. Springer, Berlin Heidelberg, 1991, pp. 385–395.

[15] R. Voorhoeve, R. de Rozario, and T. Oomen, "Identification for motion control: Incorporating constraints and numerical considerations," in *2016 American Control Conference (ACC)*, Boston, MA, USA, July 2016, pp. 6209–6214.

[16] D. S. Bayard, "High-order multivariable transfer function curve fitting: Algorithms, sparse matrix methods and experimental results," *Automatica*, vol. 30, no. 9, pp. 1439–1444, 1994.

[17] R. Vandebril, M. Van Barel, and N. Mastronardi, *Matrix Computations and Semiseparable Matrices: Eigenvalue and Singular Value Methods*. Johns Hopkins University Press, Baltimore, Maryland, 2008, vol. 2.

[18] T. Bella, Y. Eidelman, I. Gohberg, and V. Olshevsky, "Computations with quasiseparable polynomials and matrices," *Theoretical Computer Science*, vol. 409, no. 2, pp. 158 – 179, 2008.

[19] Y. Eidelman and I. Gohberg, "On a new class of structured matrices," *Integral Equations and Operator Theory*, vol. 34, no. 3, pp. 293–324, Sep 1999.

[20] L. Gemignani and L. Robol, "Fast Hessenberg reduction of some rank structured matrices," *SIAM Journal on Matrix Analysis and Applications*, vol. 38, no. 2, pp. 574–598, 2017.

[21] M. A. Beijen, R. Voorhoeve, M. F. Heertjes, and T. Oomen, "Experimental estimation of transmissibility matrices for industrial multi-axis vibration isolation systems," *Mechanical Systems and Signal Processing*, 2018.

[22] R. Vandebril, "Chasing bulges or rotations? a metamorphosis of the QR-algorithm," *SIAM Journal on Matrix Analysis and Applications*, vol. 32, no. 1, pp. 217–247, 2011.

[23] R. Vandebril, M. Van Barel, and N. Mastronardi, *Matrix Computations and Semiseparable Matrices: Linear systems*. Johns Hopkins University Press, Baltimore, Maryland, 2007, vol. 1.