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Accurate $\mathcal{H}_\infty$-Norm Estimation via Finite-Frequency Norms of Local Parametric Models

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***Abstract*— $\mathcal{H}_\infty$-norm estimation is of critical importance for robust control design. This paper aims to develop an algorithm to estimate the $\mathcal{H}_\infty$-norm with a limited amount of data and limited user intervention. A frequency response-based approach is pursued, and local modeling techniques are exploited to include the inter-grid behavior. The main idea is to estimate the global $\mathcal{H}_\infty$ norm by estimating the finite-frequency $\mathcal{L}_\infty$ norm of the local models through the generalized KYP lemma. A simulation example shows the superiority of the proposed algorithm.**

I. INTRODUCTION

Accurate $\mathcal{H}_\infty$-norm estimation is of critical importance for robust control [1], [2]. On the one hand, if the uncertainty is underestimated, there are no stability nor performance guarantees if the robust control design is applied to the true system. On the other hand, if the uncertainty is overestimated, the system may become overly conservative as the robust controller may deal with a larger class of systems than strictly necessary. In this paper, an algorithm is developed to estimate the $\mathcal{H}_\infty$ norm. The algorithm should facilitate i) reliable and accurate estimation of the $\mathcal{H}_\infty$ norm using a limited amount of data, and (ii) limited user intervention.

The availability of reliable and systematic robust control algorithms has spurred the development of algorithms to estimate the $\mathcal{H}_\infty$ norm. First, model validation techniques have been developed in [3], [4]. However, typically these methods lead to an underestimation of the $\mathcal{H}_\infty$ norm. Second, parametric modeling techniques have been developed in [5]. However, such an approach leads to a high level of user intervention to select a model structure and an appropriate model order. Third, iterative data-driven techniques have been developed in [6], [7]. Hence, a series of experiments need to be performed. However, the amount of experiments required for these iterative methods inflate when considering MIMO systems. Fourth, non-parametric approaches are developed in [8], [9]. In [8], the $\mathcal{H}_\infty$ norm is estimated by evaluating the uncertainty over a discrete frequency grid. However, in-between the discrete frequency grid, the behavior is not evaluated. Hence, it potentially leads to inter-grid errors and thus to unreliable estimates. In [9], the inter-grid error is bounded in a worst-case manner. However, these deterministic approaches typically lead to overly conservative estimates of the $\mathcal{H}_\infty$ norm [10], [11].

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In sharp contrast, local parametric models provide a golden mean between full-parametric models on the one hand and fully data-driven methods on the other hand. The key benefit of local parametric models is that the inter-grid behavior is estimated while the complex model selection step of an full-parametric model is avoided. In addition, in contrast to a full-parametric model, identifying multiple local parametric models allows parallel computing. The main idea is to exploit local smoothness of the system by local modeling techniques. Local modeling techniques are well known for non-parametric system identification techniques [12]. These techniques are being used to improve the at-grid frequency estimates which ignore the inter-grid behavior as in [8]. In [13], the potential of exploiting these local models to estimate the $\mathcal{H}_\infty$ norm is shown. Since these local models are parametric, they can be evaluated in a certain frequency interval, including the inter-grid behavior. However, in [13], resonances may be overlooked and hence no statements can be given about estimation quality.

Although important progress has been made in estimating the $\mathcal{H}_\infty$ norm using local parametric models, at present these methods lead to inaccurate $\mathcal{H}_\infty$-norm estimates. This paper aims to develop an algorithm to accurately and reliably determine the $\mathcal{H}_\infty$ norm with a limited amount of data and limited user effort. This is achieved by computing the finite-frequency $\mathcal{L}_\infty$ norm of each local parametric model in its corresponding frequency range. Accurately estimating the finite-frequency $\mathcal{L}_\infty$ norm of a local parametric model in a specific frequency interval is key to estimate the global $\mathcal{H}_\infty$ norm of a system. In [14], [15], the peak value of a system is computed on an infinite-frequency interval. Through weighting filters, the local regime can be emphasized. However, these weighting functions complicate computations and essentially approximate the problem. In this paper, an algorithm is developed to compute the finite-frequency $\mathcal{L}_\infty$ norm of a local parametric model on a finite-frequency interval. The key step is the observation that the peak value estimation problem can be reformulated to an infinite-dimensional Frequency Domain Inequality (FDI). By exploiting the generalized KYP lemma [16], [17], [18], an optimization problem in terms of a finite-dimensional LMI is obtained. The resulting algorithm enables accurate and reliable $\mathcal{H}_\infty$-norm estimation with limited data and user intervention.

The main contributions of this paper are:

(C1) the development of a new method for estimating the $\mathcal{H}_\infty$-norm by exploiting local parametric modeling techniques.

(C2) the development of a new method to compute the $\mathcal{L}_\infty$-norm in a finite-frequency window for local models.

(C3) the application of the proposed method on a relevant simulation example.

This paper is organized as follows. In Section III, the problem considered in this paper is introduced. Local modeling techniques for the global $\mathcal{H}_\infty$-norm estimation are exploited in Section IV. In Section V, an algorithm is developed to compute the global $\mathcal{H}_\infty$ norm by finite-frequency $\mathcal{L}_\infty$-norm computation of local models. In Section VI, the proposed method is applied in a simulation example. Lastly, the conclusion is provided in Section VII.

II. Notation

The following notation is used throughout. For a matrix M, the transpose and complex conjugate transpose are denoted by $M^\top$ and M^* respectively. The set $\mathbb{H}_n$ denotes the set of Hermitian matrices of size $n \times n$. For a matrix $M \in \mathbb{H}$, $M \succ (\succeq) 0$ and $M \prec (\preceq) 0$ denote positive (semi)definiteness and negative (semi)definiteness. For the matrices X, Y, $X \otimes Y$ denotes the Kronecker product. For a matrix X, $\bar{\sigma}(X)$ and $\lambda_{\max}(X)$ denote the largest singular value and largest eigenvalue.

III. Problem Formulation

A. Background

Robust control amounts to explicitly addressing uncertainty in controller design. A crucial step is to identify a model set $\mathcal{P}$ which encompasses the true system P_o. This allows robust control to provide guarantees on stability and performance for the robust controller when applied to the true system. Clearly, these guarantees hinge on accurate estimation of the model set $\mathcal{P}$. Such a model set is identified by identifying a parametric nominal model $\hat{P}$ and subsequent uncertainty estimation $\Delta(\hat{P}, P_o)$. The uncertainty can be interpreted as the model-reality mismatch. Typically, a model set is selected based on an $\mathcal{H}_\infty$-norm bounded uncertainty

$$\mathcal{P} = \left\{ P \in \mathbb{C}^{n_y \times n_u} \middle| \|\Delta(\hat{P}, P)\|_\infty \leq \gamma \right\} \tag{1}$$

where the structure of $\Delta(\hat{P}, P)$ depends on design choices made by the control engineer. An example of an uncertainty structure is the additive structure $\Delta = \hat{P} - P$. In any case, identifying a suitable model set $\mathcal{P}$ requires accurate identification of the $\mathcal{H}_\infty$-norm bound γ. Estimating an appropriate value of γ is essential for the performance of the resulting robust controller. Indeed, if the value γ is underestimated, the resulting closed-loop system may become unstable. If the bound γ is overestimated, the resulting robust controller may be conservative.

A complicating aspect is that the true system is not known explicitly, i.e., only finite-time noise-corrupted experiments can be performed.

B. Problem Formulation

In this paper, a frequency domain-based approach is pursued to determine the $\mathcal{H}_\infty$ norm. Consider the frequency domain interpretation of the $\mathcal{H}_\infty$ norm.

Definition 1: Given an asymptotically stable linear time-invariant system Δ. Then, the $\mathcal{H}_\infty$ norm is defined as

$$\gamma = \|\Delta\|_\infty = \sup_{\xi \in \Omega} \bar{\sigma}\left(\Delta(\xi)\right) \tag{2}$$

where Ω denotes the set of frequencies which becomes when formulated in Laplace domain $\Omega = j\omega$, $\omega \in \mathbb{R}$ and in Z-domain $\Omega = e^{j\omega}$, $\omega \in [0, 2\pi)$.

During experimentation, only a finite set of noise-corrupted data can be captured, consequently, the conventional non-parametric approaches give only a finite and discrete set of frequencies [8]. As a result, the discrete frequency grid may not capture the frequency at which the peak value occurs

$$\tilde{\omega} = \arg\max_{\xi \in \Omega} \bar{\sigma}\left(\Delta(\xi)\right). \tag{3}$$

Hence, potential resonances may be missed by conventional non-parametric methods which potentially causes underestimates. In this paper, an algorithm is developed to obtain an accurate and reliable estimate of the $\mathcal{H}_\infty$ norm while using a limited amount of data.

C. Approach

The main idea in this paper is to apply local parametric modeling techniques to estimate the global $\mathcal{H}_\infty$ norm. Since local models are only valid in a finite-frequency interval, the frequency axis is partitioned into continuous segments Ω_k. The key advantage of local parametric modeling is that the inter-grid behavior can be estimated which enables accurate peak value estimates. Note that local parametric models are proposed in [12]. However, in [12], local models are used to enhance the at-grid estimation quality. In contrast, in this paper, local models are used to estimated the inter-grid behavior.

The $\mathcal{H}_\infty$ norm is estimated in two steps, the first step is to compute the finite-frequency $\mathcal{L}_\infty$ norm of each local model in its corresponding frequency interval

$$\gamma_k = \sup_{\xi \in \Omega_k} \bar{\sigma}\left(\tilde{\Delta}_k(\xi)\right) \tag{4}$$

where $\tilde{\Delta}_k$ denotes the k-th local model. The second stage is to determine the global $\mathcal{H}_\infty$ norm by taking the maximum over the local peak values

$$\tilde{\gamma} = \max_k \gamma_k. \tag{5}$$

Determining the finite-frequency $\mathcal{L}_\infty$ norm is an essential step in finding the global $\mathcal{H}_\infty$ norm. Since the local models are parametric, they can be evaluated continuously over their local frequency interval. In this paper, an algorithm is developed which reliably and accurately determines the finite-frequency $\mathcal{L}_\infty$ norm of local models. This is achieved by reformulating the finite-frequency $\mathcal{L}_\infty$-norm computation problem to an LMI-based optimization problem by exploiting generalized KYP lemma.

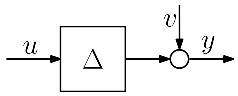

Fig. 1. Open-loop additive noise block scheme.

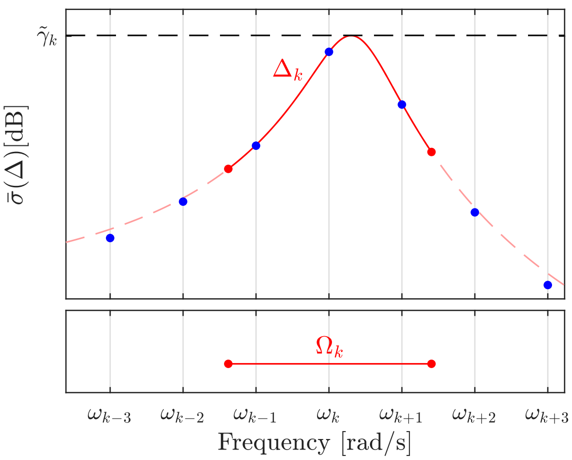

Fig. 2. Illustration of a local model $\tilde{\Delta}_k$ (top, —) with validity range Ω_k (bottom, —) estimated to the Frequency Respone Function (FRF) of Δ (•).

IV. Local Modeling for $\mathcal{H}_\infty$-norm Estimation

Accurately describing the inter-grid behavior is crucial to obtain an accurate and reliable estimate of the $\mathcal{H}_\infty$ norm. This section aims to exploit local modeling techniques to estimate the inter-grid behavior, which constitutes Contribution C1. Consider the discrete signal $u(n)$, $n = 0, 1, ..., N-1$. The discrete Fourier transform is defined as

$$U(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} u(n) \exp(\frac{-i2\pi kn}{N})$$

Consider the linear-time-invariant system in Fig. 1 where $y(n)$ denotes the output signal and $v(n)$ denotes the output noise. The output noise $v(n)$ is coloured noise in the sense that $v(n) = H(q)e(n)$ where $H(q)$ is the noise model and $e(n)$ is white noise. The response of the system in Fig. 1 with respect to the discrete input $u(n)$ in frequency domain equals

$$Y(k) = \Delta(\xi_k)U(k) + V(k) + T_{\tilde{\Delta}}(\xi_k) \tag{6}$$

where the transient response of the system Δ and noise H are lumped in the term $T_{\tilde{\Delta}}(k)$. The variable ξ_k denotes the frequency variable which becomes when formulated in Laplace domain $\xi_k = j\omega_k$ and in Z-domain $\xi_k = e^{j\omega_k}$. The key mechanism of local modeling approaches is to exploit the local smoothness of the system. This is achieved by identifying a model with validity over a finite-frequency interval. The key benefit of local modeling techniques for $\mathcal{H}_\infty$-norm estimation is that these local models are parametric. As a consequence, they can be evaluated continuously in their local frequency interval. To achieve this, a local model $\tilde{\Delta}_k$ is fitted by considering a frequency interval around the k-th frequency bin denoted by r which ranges from $-N_W$ to N_W

$$Y(k+r) = \tilde{\Delta}_k(r)U(k+r) + \tilde{T}_{\tilde{\Delta},k}(r) \tag{7}$$

where $\tilde{\Delta}_k(r)$ and $\tilde{T}_{\tilde{\Delta},k}(r)$ denote the k-th local model of the system and the transient respectively. An illustration of a local model is shown in Fig. 2.

A. Models for $\mathcal{H}_\infty$-norm estimation

The local models $\tilde{\Delta}_k$ are parameterized as local rational functions. The local rational models locally approximate Δ by estimating a rational function for each frequency bin k in a local window with $r = -N_W, ..., N_W$

$$\tilde{\Delta}_k(r) = D_k^{-1}(r)N_k(r), \tag{8}$$

$$T_{\tilde{\Delta},k}(r) = D_k^{-1}(r)M_k(r), \tag{9}$$

where $D_k(r) \in \mathbb{C}^{n_y \times n_y}$, $N_k(r) \in \mathbb{C}^{n_y \times n_u}$ and $M_k(r) \in \mathbb{C}^{n_y \times 1}$ denote the common denominator matrix, system numerator matrix and the transient numerator vector respectively

$$D(r) = I + \sum_{s=1}^{N_D} D_s(k)r^s, \tag{10}$$

$$N(r) = \sum_{s=0}^{N_N} N_s(k)r^s, \tag{11}$$

$$M_k(r) = \sum_{s=0}^{N_M} M_s(k)r^s. \tag{12}$$

The order of $D_k(r)$, $N_k(r)$ and $M_k(r)$ is denoted by N_D, N_N and N_M respectively, see [19] for this parameterization. The parameterization also includes polynomial models by selecting $N_D = 0$. The model parameters in (10), (11) and (12) can be determined through dedicated algorithms [19].

Each local model $\tilde{\Delta}_k(r)$ has validity over a finite-frequency interval

$$\Omega_k = [r_{\min,k} \quad r_{\max,k}], \tag{13}$$

where the parameters $r_{\min,k}$ and $r_{\max,k}$ determine the amount of overlap of the local model $\tilde{\Delta}_k$ with its adjacent local models. The value of $r_{\min,k}$ and $r_{\max,k}$ are determined by the control engineer. The main idea is that the local models are evaluated over a continuous yet finite interval to include the inter-grid behavior in the estimation of the $\mathcal{H}_\infty$ norm. In sharp contrast, in [12], [20], [19], local models are solely used to improve the at-grid estimate.

V. Finite-Frequency $\mathcal{L}_\infty$ Norm of Local Models

A key aspect in the estimation of the global $\mathcal{H}_\infty$ norm is to determine the finite-frequency $\mathcal{L}_\infty$ norm of the local models in their local frequency interval, e.g. solving (4) for each frequency bin k. As the local models are parametric, they can be evaluated continuously in their frequency interval. In this section, an algorithm is developed to compute the finite-frequency $\mathcal{L}_\infty$ norm of local parametric models which constitutes Contribution C2.

A. Towards finite-frequency $\mathcal{L}_\infty$-norm computation

A key step is the observation that the peak-value estimation problem can also be formulated as a validation problem in terms of a finite-frequency FDI.

Lemma 1: Suppose that $\tilde{\Delta}_k(r)$ is a local rational model which is valid on the domain Ω_k defined by (13). Then, the following statements are equivalent.

1) The finite-frequency $\mathcal{L}_\infty$ norm of $\tilde{\Delta}_k(r)$ on the domain Ω_k is bounded by γ_k

$$\sup_{r\in\Omega_k} \bar{\sigma}\left(\tilde{\Delta}_k(r)\right) < \gamma_k. \tag{14}$$

2) The FDI

$$\begin{bmatrix}\tilde{\Delta}_k(r)\\ I\end{bmatrix}^* \Pi \begin{bmatrix}\tilde{\Delta}_k(r)\\ I\end{bmatrix} \prec 0 \tag{15}$$

holds for all $r \in \Omega_k$, where Π contains the finite-frequency $\mathcal{L}_\infty$-norm bound γ_k

$$\Pi = \begin{bmatrix} I & 0\\ 0 & -\gamma_k^2 I\end{bmatrix}. \tag{16}$$

Equivalence between (14) and (15) is proved by the equality $\bar{\sigma}^2(\tilde{\Delta}_k(r)) = \lambda_{\max}(\tilde{\Delta}_k(r)\tilde{\Delta}_k^*(r))\ \forall r \in \Omega_k$.

Essentially, Lemma 1 establishes an equivalence between (14) and the FDI in (15). Hence, the FDI in (15) can be exploited to estimate the finite-frequency $\mathcal{L}_\infty$ norm in (4). However, the FDI is infinite dimensional, i.e. an infinite set of inequalities need to be verified. One approach to approximately verify an FDI is to grid the frequency axis and check whether (15) holds for all each frequencies $r \in \Omega_k$ in the discrete frequency grid. However, the FDI may not hold between the sample points. This section aims to develop a new algorithm to compute the finite-frequency $\mathcal{L}_\infty$ norm based on a finite-dimensional LMI.

A second important step in the development of an optimization problem in terms of a finite-dimensional LMI is the observation that a frequency range can be expressed as a curve on the complex plane. In this paper, a specific class of curves in the complex plane is considered based on the quadratic form

$$\Lambda(\Phi,\Psi) = \left\{\xi \in \mathbb{C} \left| \begin{bmatrix}\xi\\1\end{bmatrix}^* \Phi \begin{bmatrix}\xi\\1\end{bmatrix} = 0, \begin{bmatrix}\xi\\1\end{bmatrix}^* \Psi \begin{bmatrix}\xi\\1\end{bmatrix} \geq 0 \right.\right\} \tag{17}$$

for the matrices $\Phi, \Psi \in \mathbb{H}_2$. The set $\Lambda(\Phi,\Psi)$ represents a curve in the complex plane if certain conditions on the matrices Φ, Ψ are satisfied, see [16] for details.

The local rational models considered in this paper are valid in a certain frequency range. In particular, the local models are valid on a segment of the real axis which is defined by Ω_k in (13). By appropriate selection of Φ and Ψ, the set $\Lambda(\Phi,\Psi)$ represents Ω_k.

Corollary 1: Let Ω_k be a segment of the real axis of the complex plane defined by (13). Then $\Lambda(\Phi,\Psi)$ represents Ω_k by selecting Φ and Ψ as

$$\Phi = \begin{bmatrix} 0 & j\\ -j & 0\end{bmatrix}, \tag{18}$$

$$\Psi = \begin{bmatrix} -1 & \frac{r_{\min,k}+r_{\max,k}}{2}\\ \frac{r_{\min,k}+r_{\max,k}}{2} & -r_{\min,k}r_{\max,k}\end{bmatrix}. \tag{19}$$

Essentially, Corollary 1 establishes an equivalence between Ω_k defined by (13) and $\Lambda(\Phi,\Psi)$ by defining Φ as (18) and Ψ as (19).

B. Finite-frequency $\mathcal{L}_\infty$-norm algorithm

In this subsection, the finite-frequency $\mathcal{L}_\infty$-norm algorithm is presented, which constitutes Contribution C2.

Theorem 1: Let $\tilde{\Delta}_k$ be a local model of the form (8) with validity range Ω_k according to (13) and let $\Lambda(\Phi,\Psi)$ be specified by (18) and (19). Suppose that $\tilde{\Delta}_k$ has the following form

$$\tilde{\Delta}_k(r) = C\left(rE - A\right)^{-1}B + D \tag{20}$$

where $A, E \in \mathbb{C}^{n_x\times n_x}$, $B \in \mathbb{C}^{n_x\times n_u}$, $C \in \mathbb{C}^{n_y\times n_x}$ and $D \in \mathbb{C}^{n_y\times n_u}$. Then, the following statements are equivalent.

1) The finite-frequency $\mathcal{L}_\infty$ norm of $\tilde{\Delta}_k(\xi)$ on the domain Ω_k is bounded by $\tilde{\gamma_k}$

$$\sup_{\xi\in\Omega_k} \bar{\sigma}\left(\tilde{\Delta}_k(\xi)\right) < \tilde{\gamma_k}. \tag{21}$$

2) There exist a $P, Q \in \mathbb{H}_n$ such that

$$Q \succ 0 \tag{22}$$

$$\begin{aligned} F(\gamma_k) = &\begin{bmatrix} A & B\\ E & 0\end{bmatrix}^* (\Phi\otimes P + \Psi\otimes Q)\begin{bmatrix} A & B\\ E & 0\end{bmatrix}\\ &+ \begin{bmatrix} C & D\\ 0 & I\end{bmatrix}^* \begin{bmatrix} I & 0\\ 0 & -\tilde{\gamma_k}^2 I\end{bmatrix}\begin{bmatrix} C & D\\ 0 & I\end{bmatrix} \prec 0. \end{aligned} \tag{23}$$

The proof is based on the generalized KYP Lemma [17], [16].

Note that the class of systems in (20) includes local polynomial models, e.g. the case when $N_D = 0$ in (8).

Theorem 1 leads to a finite-dimensional feasibility test in the matrix variables P, Q for a fixed finite-frequency $\mathcal{L}_\infty$-norm bound $\tilde{\gamma}_k$. The local $\mathcal{L}_\infty$ bound γ_k in (4) is found by iterating over the bound $\tilde{\gamma_k}$ through a bi-section algorithm.

Algorithm 1: Finite-frequency $\mathcal{L}_\infty$-norm computation

- Initialization: find $[\gamma_{l,1}, \gamma_{u,1}]$ such that $F(\gamma_{l,1})$ is infeasible and $F(\gamma_{u,1})$ is feasible, set $i = 1$.
- Bisection Algorithm:
  1) Set $\gamma_{k,i} = \frac{\gamma_{l,i}+\gamma_{u,i}}{2}$.
  2) If $F(\gamma_{k,i})$ is feasible, set $\gamma_{u,i+1} = \gamma_{k,i}$, $\gamma_{l,i+1} = \gamma_{l,i}$;
     Else, set $\gamma_{l,i+1} = \gamma_{k,i}$, $\gamma_{u,i+1} = \gamma_{u,i}$.
  3) If converged, stop with $\gamma_k = \frac{\gamma_{l,i}+\gamma_{u,i}}{2}$.
     Else, set $i = i + 1$ and go to step 1.

By solving the feasibility-based algorithm in Algorithm 1 for each frequency bin k, and subsequent selection of the peak upper bound according to (5), an accurate and reliable estimate is obtained of the global $\mathcal{H}_\infty$ norm of Δ.

VI. EXAMPLE

In this section, the algorithm proposed in this paper is applied in an example, which confirms the potential and constitutes Contribution C3.

A. Model Structure and Identification

In the example a MIMO system is considered with two inputs and two outputs with weak interaction. The system P_o is a 12-th-order mass-spring damper system which is a representative example for motion systems [21]. Fig. 3 depicts a Bode diagram of the considered system. The system P_o is stabilized by the experimental controller C^{exp} which is a diagonal PD controller that delivers a reasonable performance. The system is sampled with a sampling frequency of 1 kHz. Furthermore, an eighth-order control-relevant nominal model $\hat{P}$ (Fig. 3) is estimated based on the algorithm in [22].

In the example, a specific uncertainty structure is considered, the robust-control relevant uncertainty structure (see Fig. 4)[22]. The model set is given by

$$\mathcal{P}^{RCR} = \left\{ P \middle| P = \left(\hat{N} + D_C\Delta\right)\left(\hat{D} + N_C\Delta\right)^{-1}, \|\Delta\|_\infty \leq \gamma \right\} \quad (24)$$

where the pair $\{\hat{N}, \hat{D}\}$ is a right coprime factorization of the robust-control-relevant nominal model $\hat{P}$, the pair $\{N_C, D_C\}$ is a right coprime factorization of the experimental controller C^{exp} and the $\mathcal{H}_\infty$ norm bound γ is to be estimated by the algorithm proposed in this paper. Since the nominal model $\hat{P}$ does not capture the complete behavior of the true system P_o, a part of the dynamics is captured in the uncertainty Δ. To estimate Δ, access to the signals Δu and Δy is needed. In this example, Δu and Δy are measured directly. It is however emphasized that direct access is not required, since Δu and Δy can be obtained directly from u and y [23].

B. Result

To estimate $\|\Delta\|_\infty$ a frequency domain-based approach is pursued. The input Δu is a random phase multisine with a linear frequency distribution with a resolution of 3.5 Hz. An additive output noise (Fig. 1) is added such that the signal-to-noise ratio is approximately 11. The local models are parameterized using the full-polynomial form with $N_D = N_N = 8$ and $N_M = 4$ according to (8) with the range set to $N_W = 12$. The validity range of the local models is set to $r_{\min,k} = -1$ and $r_{\max,k} = 1$.

In Fig. 5, the maximum singular values of the true uncertainty and the resulting local models are depicted for a large frequency range. In Fig. 6, a subset of the frequency range is depicted to indicate the peak-singular values and the inter-grid behavior of the local models. When studying the interpolation performance of the local models, it is clear that, in general, the true inter-grid behavior is accurately modeled. However, at low frequencies, the local models deviate from the true uncertainty. This is caused by the coarse frequency grid at low frequencies due to the linear frequency distribution in the multisine. At higher frequencies, where the grid is dense, the true system behavior is captured accurately.

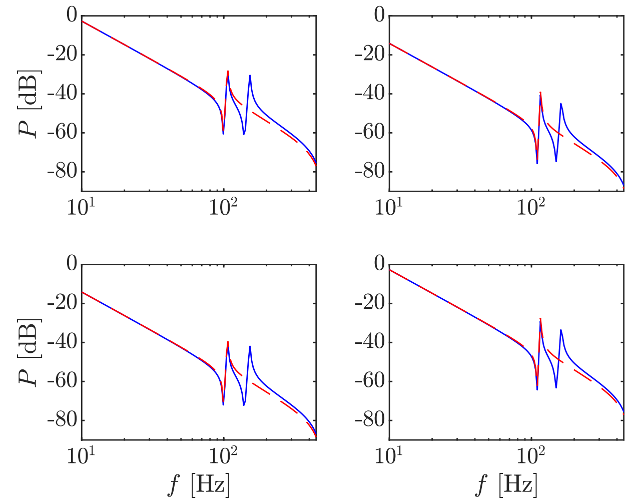

Fig. 3. Bode magnitude diagram: true system P_o (——) and parametric nominal model $\hat{P}$ (- - -).

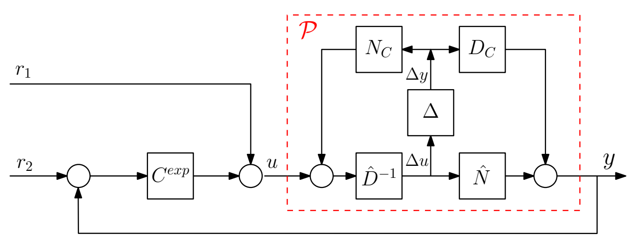

Fig. 4. Robust-control-relevant uncertainty structure in closed loop.

TABLE I

$\mathcal{H}_\infty$-NORM ESTIMATION COMPARISON.

	$\mathcal{H}_\infty$ [dB]	Error [dB]
True	82.50	-
This paper (inter-grid)	82.48	0.02
Traditional (at-grid)	78.61	3.89

When evaluating the finite-frequency $\mathcal{L}_\infty$-norm estimation of the local models in Fig. 6, it is clear that the algorithm accurately computes the peak-singular value γ_k in each validity range Ω_k. As a result the $\mathcal{H}_\infty$-norm estimate $\tilde{\gamma}$ is accurate (Table VI-B) and deviates only 0.02 dB of the true bound γ. In sharp contrast, if the $\mathcal{H}_\infty$ norm is estimated solely based on the at-grid behavior, an $\mathcal{H}_\infty$-norm estimate is obtained which deviates 3.89 dB from the true uncertainty. Hence, the simulation example shows that the method proposed in this paper offers an accurate and reliable approach to estimate $\mathcal{H}_\infty$ norm with a limited amount of data and a limited amount of user intervention.

VII. CONCLUSIONS

$\mathcal{H}_\infty$-norm estimation is of critical importance for high-performance robust control design. In this paper, an algorithm is developed which estimates the $\mathcal{H}_\infty$ norm accurately and reliably using a limited amount of data and limited

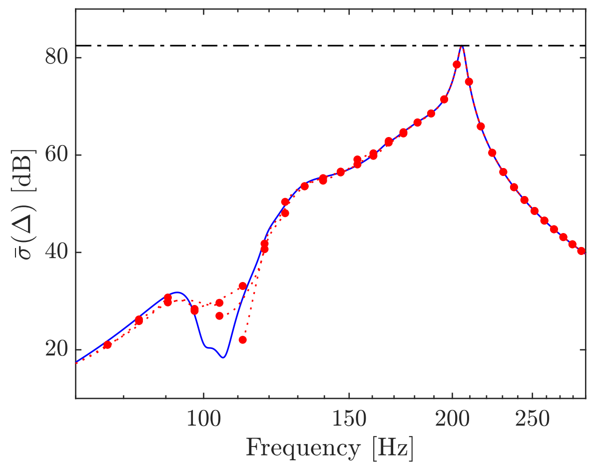

Fig. 5. Bode magnitude diagram: true uncertainty $\bar{\sigma}(\Delta)$ (—), local modeling estimate $\bar{\sigma}(\tilde{\Delta}_k)$ (• · ·•).

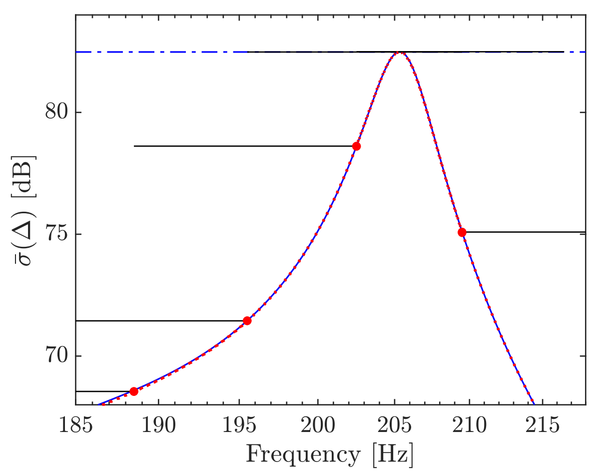

Fig. 6. Bode magnitude diagram: true uncertainty $\bar{\sigma}(\Delta_o)$ (—), local modeling estimate $\bar{\sigma}(\tilde{\Delta}_k)$ (· · ·), developed finite-frequency $\mathcal{L}_\infty$-norm estimate γ_k (−) and true $\mathcal{H}_\infty$ norm γ(−·−).

user intervention. A frequency response-based approach is pursued. Unlike traditional frequency response-based $\mathcal{H}_\infty$-norm estimation algorithms which only include the at-grid behavior, the algorithm proposed in this paper also includes the inter-grid behavior. This is achieved by exploiting local models which establish a golden mean between full-parametric modeling and non-parametric modeling. Since local models are valid in a finite-frequency range, the global $\mathcal{H}_\infty$ norm is estimated by computation of finite-frequency $\mathcal{L}_\infty$-norms of the local models. To compute the finite-frequency $\mathcal{L}_\infty$-norm, a new algorithm is developed based on the generalized KYP lemma. The effectiveness of the proposed algorithm is shown in a relevant simulation example.

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