OPTIMAL DIGITAL CONTROL OF ANALOG SYSTEMS: A SURVEY

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Abstract: This paper deals with the design of digital controllers for continuous time systems. A currently applied model-based control design approach, based on the notion of signal and system norms, computes an optimal continuous time controller that is discretized *a posteriori*. In this case, the achievement of certain performance specifications can only be guaranteed for an infinitely high sampling frequency. Two alternative approaches are proposed: one deals solely with discrete time signals, whereas the other involves both continuous and discrete time signals. In both cases, it can be guaranteed that the performance specifications are achieved. This can result in higher performance since the explicit controller discretization step is not required. The implications of both approaches regarding the entire control design procedure are discussed.

Keywords: sampled-data systems, loopshaping, intersample behavior, optimal control, MIMO control

1. INTRODUCTION

The goal of a control system is to improve the performance of a system or to let it work at all, *i.e.*, to control the output of a plant in a desired way by manipulating its input. Performance must be achieved despite uncertainties, disturbances, and measurement errors. Traditionally, such control systems were implemented as analog systems consisting of resistors, capacitors, *etc.* This concept is depicted in Fig. 1.

In the last decades, there has been a huge development in the field of digital computers (Åström and Wittenmark, 1990). Many analog control systems have been replaced by digital ones because of lower cost, increased flexibility, and superior accuracy. In that case, the system is controlled using sampled observations of the system that are obtained by an Analog-Digital (AD) convertor. The Discrete Time (DT) control law computes an input to the plant that is converted into the Continuous Time (CT) domain by means of a hold circuit, that is, a Digital-Analog (DA) convertor. This concept is shown in Fig. 2, where the sample and hold device are part of the control system.

Because the system in Fig. 2 is hybrid, *i.e.*, it consists of interacting CT and DT components, the design of a control system is not straightforward. Current control design methods are still focused on the CT (analog) domain. After designing an analog controller, a controller discretization step

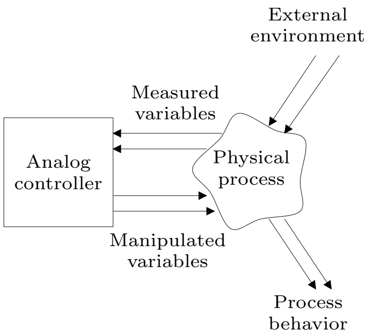

Fig. 1. Process controlled by an analog controller

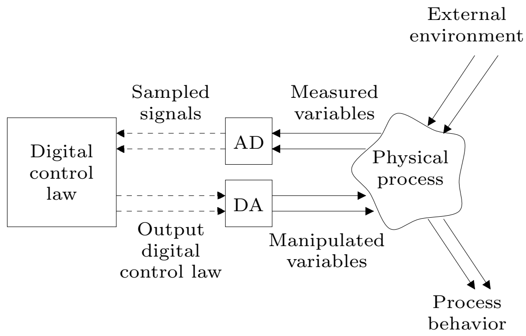

Fig. 2. Process controlled by a digital controller

is performed. It is well-known that the obtained digital controller resembles the CT controller if the sample time tends to zero. However, in practical applications the sample time is lower-bounded due to computational reasons. This can result in performance degradation and even instability of the closed-loop system.

Especially when the CT controller is tuned for very high performance, the discretization step may result in significant performance degradation. The main reason for this is due to the fact that high-performance controllers usually exhibit relevant dynamics beyond the target bandwidth of the control system. A theoretically well-founded method to design such high-performance controllers is by means of model-based optimal control. This approach can also deal with Multi-Input Multi-Output (MIMO) systems and is based on the notion of signal and system norms. Roughly speaking, optimal control methods amount to translating the control design problem into a mathematical optimization problem that resorts to a model of the true system. Because the system generally evolves in the CT domain, it is most natural to use a CT model of the system, see also Fig. 1. In that case, the optimization problem typically delivers a CT controller, see, *e.g.*, (Kwakernaak and Sivan, 1972), (Doyle et al., 1989). Especially for model-based control algorithms, it may be beneficial to take the implementation step (*i.e.*, the DA and AD convertor) into account during controller synthesis.

The remainder of this paper is organized as follows. In Sec. 2, a representative control design procedure for MIMO systems based on $\mathcal{H}_\infty$-optimization/μ-synthesis is reviewed, together with several shortcomings. In Sec. 3, two alternative approaches are discussed towards resolving these shortcomings. Then, in Sec. 4 an example is given to illustrate the importance of intersample behavior, *i.e.*, the behavior of the system between the sampling instants. Finally, in Sec. 5 several conclusions are drawn.

Throughout this paper, it is assumed that the DA convertor $\mathcal{H}$ is an ideal Zero-Order-Hold (ZOH), *i.e.*, $(\mathcal{H}(u[k]))(t) := u[k], t \in [kh, (k+1)h)$, where h denotes the sample time and the AD convertor $\mathcal{S}$ is the ideal sampler, *i.e.*, $\mathcal{S}(u)[i] := u(kh)$. Although better digital controllers may be designed if the DA convertor is free for design, the ZOH circuit is by far the most common in practical applications.

2. CURRENT $\mathcal{H}_\infty$ CONTROL DESIGN APPROACH

In this section, a design procedure based on $\mathcal{H}_\infty$-optimization/μ synthesis is introduced that is used at the Philips Centre for Industrial Technology (CFT). In Sec. 2.1, the standard plant is introduced that is a fundamental concept in the optimal control literature. Then, in Sec. 2.2 the current design approach that has succesfully been applied at the CFT is reviewed.

2.1 Norm-based Control

In the optimal control literature, the standard plant is often encountered, see Fig. 2.1. Most control design problems can be described by the standard plant, *e.g.*, the classical one-degree-of-freedom feedback configuration. In this standard plant, the influences of the external environment are collected in a vector of exogenous inputs $w(t)$, *e.g.*, disturbances and reference signals. The variables that represent the control objectives are collected in a vector $z(t)$ and are defined such that they should be minimized. Typically, $z(t)$ contains error and actuator signals. The measured variables are collected in a vector $y(t)$, whereas the manipulated variables are contained in $u(t)$. In Fig. 2.1, G denotes the standard plant that contains the plant model, weighting filters, and an interconnection structure. For notational convenience, model uncertainty is disregarded at this point. The goal of optimal control is to minimize the system norm of $M : w \mapsto z$.

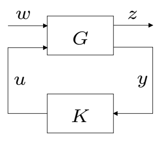

Fig. 3. Standard plant

2.2 Current Design Steps

In this section, the design steps of the current approach applied at CFT are given. For a more detailed discussion, see (Van de Wal et al., 2002). The following steps are performed [1] :

i) Control goal and control structure: define the exogenous outputs $z(t)$, *i.e.*, the variables that should be kept small in the controller system. Also, define the exogenous inputs $w(t)$, *i.e.*, the signals that enter the controlled system. Moreover, define the signals that are measured ($y(t)$) and manipulated ($u(t)$), *i.e.*, the signals that are available as input and output of the controller, respectively.

ii) Model the plant as a CT finite dimensional Linear Time Invariant (LTI) system using System Identification techniques (SI) (Ljung, 1999) and reduce its order if necessary (Wortelboer, 1994). In this paper, the SI techniques are restricted to the frequency domain, since this has several clear advantages over time domain identification, see Pintelon et al. (1994).

iii) Quantify the control goals by means of frequency domain weighting filters. The proposed weighting filters in (Van de Wal et al., 2002) resemble manual loopshaping ideas. In particular, several closed-loop transfer functions such as the sensitivity and the complementary sensitivity function are shaped. These filters are absorbed into the standard plant G.

iv) Quantify the model uncertainty. This step is optional and allows the control designer to explicitly incorporate an uncertainty model into the control design procedure.

v) Controller synthesis and controller order reduction. Synthesize the controller based on the generalized plant, using a `Matlab` toolbox (Balas et al., 2001). If the order of the resulting controller is too high, reduce its order to enable real-time implementation (Wortelboer, 1994).

vi) Discretize the controller. Since the model derived in Step ii is stated in the CT domain, the controller resulting from Step v is typically a CT controller that must be discretized for implementation in a digital environment.

2.3 Problems Current Approach

In the current approach, performance degradation may be introduced if the designer switches between the CT and DT domain in the control design procedure. This takes place in Steps ii and vi. The discretization in Step vi always involves an approximation, see Åström and Wittenmark (1990) and Franklin et al. (1998) for an overview of different discretization methods. Hence, it is desirable to avoid this step in the control design procedure since it generally results in performance degradation. Another problem involves Step ii, where a CT model is fitted on a measured Frequency Response Function (FRF). This FRF is measured under DT circumstances, *i.e.*, it describes the transfer function between an input sequence $\nu[k]$ and an output sequence $\psi[k]$, see Fig. 4. This implies that the frequency domain action of the hold circuit (that has a low-pass characteristic and a delay of half a sample time) is taken into account up to a certain extent in the CT model. Although this may be beneficial in the control design procedure, the fitted model does not describe the true CT system $P(s)$ well. In the literature, this strategy is sometimes referred to as Pseudo Continuous Time (PCT) (Houpis and Lamont, 1985).

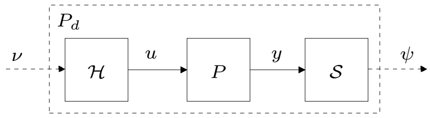

Fig. 4. Relation CT plant (P), DT plant (P_d), sampler $\mathcal{S}$, and hold circuit $\mathcal{H}$

3. IMPROVEMENTS

In this section, several approaches are presented that improve or avoid the transition between the CT and DT domain, such that higher performance may be achieved by model-based control designs.

3.1 Modeling

When modeling a system using system identification techniques, the designer almost always relies on sampled data obtained from the plant. These sampled signals do not carry over all the information of the true CT signals, hence the modeling of the CT system $P(s)$ requires certain assumptions regarding the properties of the signals. As an alternative, the DT system $P_d(z)$ can be identified. Both approaches are elaborated on next.

3.1.1. DT Modeling Probably the most straightforward method to model the system in Fig. 4 is to consider it as a purely DT system that maps a sequence $\nu[k]$ into $\psi[k]$. The signal $\nu[k]$ can be considered as a sampled CT signal. For instance, assume that the CT signal to be sampled is a sinusoidal signal

$$r(t) = A\sin(\omega t + \phi_1), \tag{1}$$

[1] In most cases, the control design is performed iteratively, several steps may be repeated.

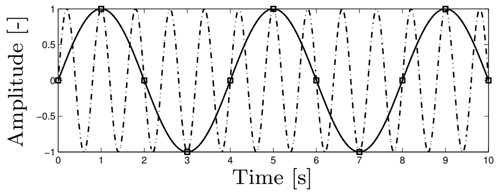

Fig. 5. Sine ($f = 0.25$ $[Hz]$) (solid), sine ($f = 1.25$ $[Hz]$) (dash-dotted), sampled sine (square)

hence

$$\nu[k] = A\sin(\omega kh + \phi_1). \quad (2)$$

If $P_d(z)$ is LTI, then it can be shown that

$$\psi[k] = B\sin(\omega kh + \phi_2). \quad (3)$$

Thus, the frequency response of $P_d(z)$ is given by $|P_d(e^{j\omega h})| = \frac{B}{A}$ and $\angle P_d(e^{j\omega h}) = \phi_2 - \phi_1$. Note that the frequency response is periodic with the sampling frequency $\omega_s = \frac{2\pi}{h}$. This can easily be understood since several sinusoidal signals have the same value at the sampling instants, *i.e.*,

$$\begin{aligned}\nu[k] &= A\sin(\omega kh + \phi_1)\\ &= A\sin((\omega + l\omega_s)kh + \phi_1),\ l \in \mathbb{Z}\end{aligned} \quad (4)$$

An example is given in Fig. 5, where $h = 1$ [s]. Concluding, a model $P(z)$ can be estimated from the DT sequences $\nu[k]$ and $\psi[k]$. Moreover, it can be shown that a DT model $P_d(z)$ exists that has the same order as the underlying CT system $P(s)$.

3.1.2. CT Modeling The designer can also model the CT system $P(s)$ in Fig. 4. For CT model fitting algorithms to be applicable, the designer should have knowledge of both $u(t)$ and $y(t)$. If $\nu[k]$ is chosen by the designer, $u(t)$ can easily be computed as a piecewise constant signal (that is, it consists of superposed step signals). The next step is to reconstruct $y(t)$ from the sampled observations $\psi[k]$. This is not a straightforward operation, since information may be lost by sampling. Whether or not $y(t)$ can be reconstructed from $\psi[k]$ is given by the next theorem, see, *e.g.*, Franklin et al. (1998).

Theorem 1. (Shannon's sampling theorem). A CT signal can be uniquely recovered from its samples if it has no frequency content above half the sample frequency, *i.e.*, the Fourier transform of the CT signal is zero outside the inverval $\omega \in [-\frac{\omega_s}{2}, \frac{\omega_s}{2}]$.

The condition in Theorem 1 is not automatically met in the setup of Fig. 4. This is best explained by the fact that $u(t)$ consists of step signals, hence it contains an infinite number of high frequencies. Since $P(s)$ is LTI, these components also appear in $y(t)$. If the designer is able to filter out these high frequent components before sampling the signal, $y(t)$ may be recovered for frequencies $|\omega| \leq \frac{\omega_s}{2}$, see Theorem 1. From $u(t)$ and $y(t)$, the CT system $P(s)$ can then be identified for $|\omega| \leq \frac{\omega_s}{2}$.

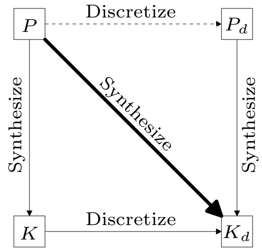

Fig. 6. Design of a digital controller for a CT plant

3.1.3. Concluding Remarks Identifying a DT model $P_d(z)$ is useful if the plant model is derived at the same sampling frequency at which the resulting controller should be implemented. Identifying the CT plant generally requires a relatively high sampling frequency compared to the plant dynamics. The identification of a CT plant can, for example, be performed by using a dedicated spectrum analyzer. Several advantages of CT identification are given in Rao and Sinha (1991), probably the most important being that the laws for physical systems are stated in the CT domain. Another advantage of CT modeling is that a DT model (that is only valid for a specific sample time) can easily be computed if the CT plant $P(s)$, the sampler $\mathcal{S}$, and the hold circuit $\mathcal{H}$ are given.

3.2 Theoretical Frameworks

In this section, two frameworks are discussed that directly result in a DT controller. The first method is discussed in Sec. 3.2.1 and requires a model $P_d(z)$ of the DT system. Hence, this method begins at the upper right block in Fig. 6. The second approach requires a CT model $P(s)$ of the physical process and begins in the upper left block of Fig. 6 and follows the diagonal, thick line. This approach is called the sampled-data (SD) approach. The approach discussed in Sec. 2.2 also starts with a (pseudo) CT model and follows the left and bottom arrow subsequently.

3.2.1. DT Framework A possible approach to design a DT controller is to identify the plant as discussed in Sec. 3.1.1, *i.e.*, to directly identify a DT model of the system that describes the input-output behavior at the sampling instants. Thus, the designer starts at block P_d in Fig. 6. Regarding the procedure of Sec. 2.2, Step ii is now performed in the DT domain. The subsequent steps are then naturally also performed in the DT domain and Step vi can be omitted.

It is important to remark that the intersample behavior is completely disregarded in this approach. Of course, it cannot be expected that a controller operating at a low sampling frequency attenuates high frequent disturbances (in particular, at the k^{th} sample instant a constant control input has to be chosen for the time interval $t \in [kh, (k+1)h)$, in this time interval the system is actually in open-loop). However, a poorly chosen control law can deteriorate the intersample behavior, as will be shown in Sec. 4. In general, it can be guaranteed that certain performance specifications are met on the sampling instants. Moreover, it can be shown that the DT controller that stabilizes the DT plant $P_d(z)$ also stabilizes the underlying CT plant $P(s)$ (Kalman et al., 1963) if observability and controllability are not lost during sampling.

3.2.2. SD Framework If a CT plant model is derived as in Sec. 3.1.2, a similar design strategy can be adopted as in Sec. 2.2. However, if the effect of the DA circuit is completely neglected, it cannot be expected that the resulting controller achieves the same performance as the controller obtained with the PCT approach, see Sec. 2.3. Several years ago a theoretical framework is developed that allows the direct design of a DT controller for a CT plant, based on the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm of an SD system, see Bamieh and Pearson (1992) and the book by Chen and Francis (1995). The synthesis of an optimal SD controller is indicated by the diagonal line in Fig. 6. This is the ideal solution, since it yields a DT controller for CT performance specifications, without the need of a controller approximation step.

3.3 Controller Design

Although the theoretical frameworks discussed in the above section are appealing, the current weighting filters cannot be used directly in the DT and SD framework. This statement is motivated in this section.

3.3.1. DT Design Because the $\mathcal{H}_\infty$ norm of a DT system equals the peak singular value of a transfer function matrix, loopshaping ideas can also be applied to DT systems. In this case, the DT closed-loop transfer functions are shaped. A difficult aspect of the weighting function selection is the periodicity of the frequency response with period ω_s, since the frequency response of a DT system $P_d(z)$ is obtained by substituting $z = e^{j\omega h}$, see also Sec. 3.1.1 and Franklin et al. (1998). This also implies that the Bode gain-phase relation does not hold for DT systems, *i.e.*, the asymptotic behavior of the frequency response is lost. This can be solved by mapping the DT system in the

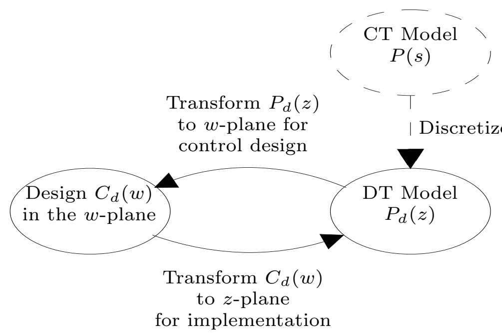

Fig. 7. Proposed DT design approach

z-plane into the w-plane (Johnson et al., 1955). Then, weighting filters can easily be defined as in the CT domain, since cut-off frequencies and asymptotes of the filters can be computed. Since this w-plane closely resembles the s-plane, CT controller synthesis algorithms (Doyle et al., 1989) can be employed in the w-plane. Since an inverse transformation exists of the mapping from the z-plane to the w-plane, the controller in the w-plane can be transformed back into the z-plane without approximation (Brown and Churchill, 1996). This idea is depicted in Fig. 7.

Remark 1. Similar techniques can be used for manual loopshaping when an FRF measurement or a DT model is given.

3.3.2. SD Design It is well-known that for SD systems the frequency-separation principle is lost. This means that if a CT exogenous signal is applied to the plant, *e.g.*, a sinusoidal disturbance at the CT plant input with frequency ω_i, the CT output of the closed-loop system consists of infinitely many frequency components, see also the discussion regarding the spectrum of $y(t)$ in Sec. 3.1.2. This implies that the concept of transfer functions does not exist for SD systems and Bode plots cannot be drawn directly.

Various definitions of the frequency response gain of an SD system [2] have appeared in the recent literature which are briefly introduced here. If a sinusoidal input of a single frequency ω_i is applied and if the CT output is considered only at that frequency, the fundamental frequency gain is obtained (Seron et al., 1997). If, instead, the power of the CT output is considered, the average power-gain matrix is obtained (Cantoni, 1998). If the input signal is chosen as the superposition of sinusoidal components with frequencies separated by an integer multiple of the sampling frequency and the maximum power norm is computed, the frequency response function of a SD system is obtained (Araki et al., 1996).

Although it can be shown that the peak value over

[2] The phase information is typically lost in the definitions.

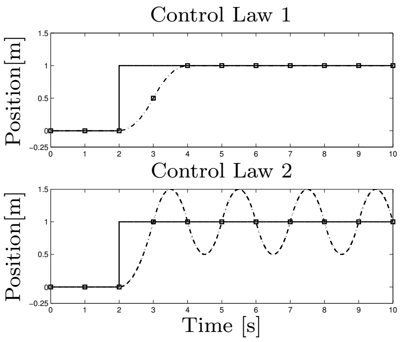

Fig. 8. Reference signal (solid), CT response $y(t)$ (dash-dotted), DT response $\psi(kh)$ (square)

frequency of the latter definition equals the $\mathcal{H}_\infty$ norm of an SD system, for performance analysis it is more useful to study the system response to sinusoidal inputs of a single frequency, *i.e.*, the fundamental frequency gain and the average power-gain matrix are more suited for loopshaping purposes. In Cantoni (1998), already an attempt was made to shape the open-loop system. It would be interesting to generalize this concept to the closed-loop case, similar to Step iii of Sec. 2.2.

4. DESIGN EXAMPLE

In this section, a design example adopted from Oomen (2004) is given to motivate the importance of intersample behavior. Optimal control methods are disregarded at this moment. Consider a CT plant given by $P(s) = \frac{1}{s^2}$, *i.e.*, a single mass of 1 [kg]. After computing the DT equivalent $P_d(z)$ where a ZOH circuit and a sample time of 1 [s] are assumed, two control laws are designed based on $P_d(z)$. The DT and CT responses $\psi[k]$ and $y(k)$ are depicted in Fig. 8. Although the DT response of Control law 2 is better, *i.e.*, a lower settling time, its intersample behavior is worse than that of Control law 1. It is concluded that performance should be evaluated in the CT domain.

5. CONCLUSIONS

In this paper, two different approaches have been discussed that avoid the explicit controller discretization step in the control design procedure and hence may result in higher performance using model-based control design approaches. Although the DT approach is most straightforward to apply in practice, it has the potential danger of poor intersample behavior. Possibly, rules of thumb can be derived regarding the choice of DT weighting filters such that the intersample behavior is not deteriorated by the control law. Alternatively, the designer may consider using the SD optimal control design procedure. The main problems with this approach are the required accurate modeling of the CT plant from sampled observations and the choice of weighting filters, since loopshaping techniques cannot be applied directly.

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