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A Robust-Control-Relevant Perspective on Model Order Selection

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***Abstract*— High-performance robust control hinges on explicit compensation of performance-limiting system phenomena. Hereto, such phenomena need to be described with high fidelity by the model set. Clearly, this demands for a delicate mutual selection of the nominal model and the uncertainty bound. Both should have a limited complexity to enable successful controller synthesis and implementation. The aim of this paper is to investigate model order selection for robust-control-relevant identification. Therefore, it is investigated how the worst-case performance that is associated with a model set is influenced by the complexity of the nominal model and the uncertainty bound. It turns out that, using a judiciously selected uncertainty coordinate frame, worst-case performance can be made invariant for the order of the uncertainty bound. Nevertheless, dynamic uncertainty modeling may still be worthwhile when accounting for approximations that are commonly made in robust-control-relevant identification, as is analyzed in this paper as well.**

I. Introduction

Establishing the appropriate level of detail is a principal aspect in the modeling of physical systems. Inevitably, any model is an approximation of reality. From this perspective, the intended application of the model determines the model quality that is needed. Control-relevant identification, see [12], focusses on accurate modeling of those system phenomena that need to be addressed explicitly in subsequent model-based control design. Indeed, performance is often limited dominantly by a restricted number of system artifacts.

Even when primary system behavior is described accurately, it is essential to account for remaining model imperfections. In fact, discrepancies between model and reality may lead to a dramatic deterioration of performance when implementing a designed controller on the true system, [7]. Robust control, [18], can cope with systematic model errors, as it accounts for a set of perturbations on the modeled behavior explicitly. Model sets for robust control should (i) encompass true system behavior, (ii) enable high-performance control, and (iii) have limited complexity. The latter requirement is essential for successful synthesis and real-time implementation of robust controllers, since the complexity of (a) the nominal model and (b) the dynamic uncertainty bound contributes to the complexity of the resulting controller, *cf.* [18].

This paper concerns system identification for robust control, applied to lightly damped, flexible systems. Important examples include [1] and [17], where a nominal model is estimated using a weighted identification criterion that aims to emphasize important system artifacts. This model is extended with an additive uncertainty set that is bounded in such a way, that true system behavior is accounted for. In [2], a further refinement in the unification of Requirement (i) and (ii) has been made by explicitly taking the robust control goal into account during construction of the model set. Hereto, a control-relevant nominal model is constructed according to [12], which is extended with a dual-Youla uncertainty set, see [15]. Again, a dynamic uncertainty bound is constructed such, that true system behavior is accounted for.

Although the selection of both the nominal model and the uncertainty bound are understood to be essential for high-performance robust control, their relative importance has remained unstudied to a large extent. Yet, a fundamental interdependence seems to exist. From an undermodeling perspective, by increasing the complexity of the nominal model, more and more true system artifacts can be described accurately. Consequently, a smaller uncertainty set suffices to encompass true system behavior by the model set. However, refinement of the nominal model is meaningless if a very crude uncertainty bound is constructed subsequently. In this paper, mutual selection of the nominal model order and the order of the uncertainty bound is analyzed, with the aim to develop insights into order selection for robust-control-relevant identification.

Order selection has been studied extensively within the prediction error framework, see [5] for an overview. However, the suggested paradigms do not comply with the robust control philosophy, since nominal modeling is considered only. The influence of the uncertainty set *has* been taken into account in the set-membership framework, where the nominal model order is chosen using the *radius of information* as a selection criterion, see, *e.g.*, [4], [8]. However, herein the complexity of the uncertainty description itself is not taken into account. Moreover, an additive uncertainty structure is assumed, although [9] specifically advocates the use of the dual-Youla uncertainty structure in identification for robust control. This paper further expands the latter approach by using unexplored freedom in the choice of coprime factorizations that constitute the dual-Youla structure. By judiciously choosing these coprime factorizations, new insights in model order selection for robust control are obtained.

The main contribution of this paper is to investigate how the complexity of the nominal model and the uncertainty bound affects the worst-case performance of the resulting model set. After some preliminaries (Sect. II) the problem of order selection for robust control is formalized (Sect. III). By exploiting freedom in the choice of coprime factorizations, *cf.* [11], that form the dual-Youla uncertainty structure (Sect. IV), the contribution of the nominal model and the uncertainty bound to worst-case performance is made transparent (Sect. V). In fact, in the considered uncertainty coordinate frame, worst-case performance is invariant for the choice of the order of the uncertainty bound. However, dynamic uncertainty modeling may still prove worthwhile, as follows from an analysis of the implications of an iterative identification and control design procedure (Sect. VI). In the last part of the paper, obtained results are connected to set-membership identification (Sect. VII-A) and illustrated on an industrial system (Sect. VIII).

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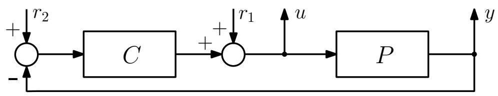

Fig. 1. Standard feedback configuration.

II. Preliminaries

A. $\mathcal{H}_\infty$-norm-based control design

Definition 1. The considered performance criterion J reads:

$$J(P,C) := \|W\,T(P,C)\,V\|_\infty\,. \tag{1}$$

Here, $T(P,C)$ maps the exogenous inputs w of the feedback configuration depicted in Fig. 1 onto the outputs z, *i.e.*:

$$T(P,C)\,:\,\underbrace{\begin{bmatrix} r_2 \\ r_1 \end{bmatrix}}_{w} \mapsto \underbrace{\begin{bmatrix} y \\ u \end{bmatrix}}_{z} = \begin{bmatrix} P \\ I \end{bmatrix} (I+CP)^{-1}\,[C \quad I]\,. \tag{2}$$

Furthermore, W and V are bistable, diagonal weights.

Control goal. Design the $\mathcal{H}_\infty$-norm optimal controller:

$$C_{\text{opt}} = \arg\min_{C} J(P_o, C), \tag{3}$$

where P_o denotes the true system.

B. Robust-control-relevant system identification

Since P_o is unknown, (3) cannot be solved explicitly. Instead, a model set $\mathcal{P}$ is constructed that envelops the true plant dynamics, *i.e.*, $P_o \in \mathcal{P}$.

Definition 2. The worst-case performance J_{WC} associated with a given $\{\mathcal{P}, C\}$ is defined as:

$$J_{\text{WC}}(\mathcal{P}, C) := \sup_{P\in\mathcal{P}} J(P,C)\,. \tag{4}$$

From Def. 2, it is immediate that (stipulating $P_o \in \mathcal{P}$):

$$J(P_o, C) \,\leq\, J_{WC}(\mathcal{P}, C)\,. \tag{5}$$

Thus, minimization of J_{WC} provides an instrument for performance optimization on the unknown true plant P_o.

Definition 3. Given $\mathcal{P}$, robust controller optimization yields:

$$C_{\text{RP}}(\mathcal{P}) := \arg\min_{C} J_{\text{WC}}(\mathcal{P}, C)\,, \tag{6}$$

Definition 4. The plant set $\mathcal{P}_{\text{RP}}$ for the synthesis of C_{RP} is obtained by solving the dual problem:

$$\mathcal{P}_{\text{RP}} := \arg\min_{\mathcal{P}} J_{\text{WC}}(\mathcal{P}, C_{\text{RP}}(\mathcal{P})) \quad \text{s.t.} \quad P_o \in \mathcal{P}. \tag{7}$$

Clearly, (7) is intractable, since it requires joint optimization of the plant set and the robust controller that is to be synthesized on the basis of this plant set. Therefore, C_{RP} is approximated by C_{exp} that can be used to conduct closed-loop identification experiments. The following robust-control-relevant identification criterion results.

Definition 5. For a given experimental controller C_{exp}, a robust-control-relevant model set $\mathcal{P}_{\text{RCR}}$ follows from:

$$\mathcal{P}_{\text{RCR}}(C_{\text{exp}}) = \arg\min_{\mathcal{P}} J_{\text{WC}}(\mathcal{P},\, C_{\text{exp}}) \quad \text{s.t.} \quad P_o \in \mathcal{P}. \tag{8}$$

To tighten (5), it may be advantageous to iterate between the identification of a robust-control-relevant model set and the design of a new robust controller, see [2].

III. Problem formulation

Deriving a tight description of true plant behavior is key to the minimization of worst-case performance in Prop. 5. The construction of a parametric model set for robust control constitutes (i) identification of a control-relevant nominal model $\hat{P}$, and (ii) quantification of an uncertainty bound $\mathcal{W}_\Delta$.

Definition 6. Let $\hat{P}^{(n)}$ denote a nominal model of order n. Let $\mathcal{W}_\Delta^{(o)}$ denote a bistable uncertainty bound of order o. The model set $\mathcal{P}$ takes the form:

$$\mathcal{P}^{(n,o)} = \Big\{\, P \;\big|\; P = \mathcal{F}_u(\,\hat{H}(\hat{P}^{(n)}),\, \Delta_u)\,,\\ \|\,\Delta_u\,(\,\mathcal{W}_\Delta^{(o)}\,)^{-1}\|_\infty < 1 \Big\}\,. \tag{9}$$

Here, $\mathcal{F}_u$ denotes the upper linear fractional transformation. By convention, $\hat{H}$ is partitioned such that $\hat{H}_{22} = \hat{P}$. The remaining blocks of $\hat{H}$ contain the uncertainty structure. The perturbations Δ_u are assumed to be unstructured.

Successful robust control design and implementation requires model sets of limited complexity, since n and o contribute to the order of the resulting controller, [18]. Hereto, criterion (8) is expanded with regularization functions, as is commonly encountered in order selection procedures, *cf.* [5].

Definition 7. Given C_{exp}, robust-control-relevant identification with regularization of the model set complexity amounts to selection of $n, o \in \mathbb{N}_0$ according to:

$$\min_{\{n,o\}} \; J_{WC}(\mathcal{P}^{(n,o)},\, C_{\text{exp}}) + f_1(n) + f_2(o) \quad \text{s.t.} \quad P_o \in \mathcal{P}^{(n,o)}. \tag{10}$$

Here, f_1, f_2 are positive, strictly monotonous regularization functions that penalize model sets of high complexity.

In the construction of parametric model sets for robust control, a fundamental trade-off exists in distributing modeling complexity over the nominal model and the uncertainty overbound. On the one hand, an increase of the nominal model order permits a more accurate characterization of true system behavior. Consequently, a smaller uncertainty set suffices for $\mathcal{P}$ to encompass true plant behavior P_o. On the other hand, if a very rough parametric description of this set is made, the gained nominal model accuracy will be nullified, since robust control needs to cope with a very broad set of admissible perturbations after all. In this paper, a paradigm is derived to trade off nominal model and overbound complexity. Herein, the dual-Youla uncertainty coordinate frame plays an essential role.

IV. Dual-Youla uncertainty structure

Stability is a key property of control-relevant model sets. Inherently, any control-relevant system identification approach accords to a closed-loop criterion [14]. To conduct identification experiments, P_o is stabilized. Consequently, true feedback behavior M_o can never be represented by an unstable model M. Such models should be excluded from the set $\mathcal{M}$ to prevent conservative robust control design.

Proposition 8. [15] Given a model $\hat{P}$ that is stabilized by a controller C. Let $\{\hat{N}, \hat{D}\}$ and $\{N_c, D_c\}$ denote (any) right coprime factorizations (RCFs) of $\hat{P}$ and C, see [18, Sect. 5.4]. The set $\mathcal{P}_{dY}$ of *all* plants that are stabilized by C is:

$$\mathcal{P}_{dY}(C) := (\hat{N} + D_c\Delta_u)(\hat{D} - N_c\Delta_u)^{-1},\; \Delta_u \in \mathcal{RH}_\infty. \tag{11}$$

Lemma 9. [2] The weighted closed-loop model set $\mathcal{M}$, obtained by substitution of $\mathcal{P}_{dY}(C)$ in Fig. 1, is given by:

$$\mathcal{M} = \left\{ M : w \mapsto z \mid M = \mathcal{F}_u(\hat{M}, \Delta_u),\ \Delta_u \in \mathcal{RH}_\infty \right\} \quad (12)$$

$$\text{with:}\quad \hat{M} = \left[\begin{array}{c|c} 0 & (\hat{D}+C\hat{N})^{-1}\,[C \quad I]\,V \\ \hline W\begin{bmatrix} D_c \\ -N_c\end{bmatrix} & W\begin{bmatrix}\hat{P}\\ I\end{bmatrix}(I+C\hat{P})^{-1}\,[C \quad I]\,V \end{array}\right].$$

The weighted true feedback loop $M_o := W\,T(P_o, C)\,V$ is stable, hence, $\exists\, \Delta_o \in \mathcal{RH}_\infty$ s.t. $M_o = \mathcal{F}_u(\hat{M}, \Delta_o)$. To facilitate robust performance optimization, perturbations Δ_u are confined to a bounded set $\mathbf{\Delta_u} \subset \mathcal{RH}_\infty$. This set is selected such that $\Delta_o \in \mathbf{\Delta_u}$. Hereto, typical uncertainty quantification procedures, including [10] and [16], deliver a nonparametric characterization of the model error magnitude, *i.e.*, $\gamma(\omega) = \bar{\sigma}(\Delta_o(\omega))$. Standard $\mathcal{H}_\infty$-norm-based controller synthesis, however, requires a *parametric* uncertainty description of the form (9). For this purpose, a bistable weight $\mathcal{W}_\Delta$ is constructed that overbounds the frequency-dependent bound $\gamma(\omega)$. The following perturbation set results.

Definition 10. The perturbation set is given by:

$$\mathbf{\Delta_u}^{(o)} \ : \ \{\Delta_u \in \mathcal{RH}_\infty \mid \|\, \Delta_u\, (\,\mathcal{W}_\Delta^{(o)}\,)^{-1}\|_\infty < 1\} \ \text{ s.t.}$$
$$\bar{\sigma}(\mathcal{W}_\Delta^{(o)}(\omega)) \geq \gamma(\omega) \ \ \forall \ \ \omega. \quad (13)$$

Commonly, dynamic uncertainty bounds $\mathcal{W}_\Delta^{(o)}$ are considered in literature, see, *e.g.*, [1],[3], [17]. However, at present, the role of the overbound order o in worst-case performance optimization is not well-understood.

V. Order selection for robust control

In this section, it is investigated how the worst-case performance associated with a model set is affected by the choice for the model order of the nominal model and the uncertainty bound. As it turns out, the particular coprime factorization that is chosen plays an important role herein.

A. Generic coprime factorizations

For generic coprime factorizations in the dual-Youla structure, the following bound on J_{WC} holds.

Lemma 11. The worst-case performance associated with $\mathcal{P}_{dY}$, see Prop. 8, is upper bounded by:

$$J_{\text{WC}}(\mathcal{P}_{dY}^{(n,o)}, C_{\text{exp}}) \ = \sup_{\Delta_u \in \mathbf{\Delta_u^{(o)}}} \|\mathcal{F}_u(\hat{M}(\hat{P}^{(n)}), \Delta_u)\|_\infty$$
$$\leq \ J(\hat{P}^{(n)}, C) + \sup_{\Delta_u \in \mathbf{\Delta_u}^{(o)}} \|\hat{M}_{21}\Delta_u\hat{M}_{12}\|_\infty. \quad (14)$$

Proof: By virtue of (12), $\hat{M}_{11} = 0$. Hence:

$$\mathcal{F}_u(\hat{M}, \Delta_u) = \hat{M}_{22} + \hat{M}_{21}\Delta_u\hat{M}_{12}. \quad (15)$$

Application of the triangle inequality and observing that $\|\hat{M}_{22}\|_\infty = J(\hat{P}, C)$ yields the desired result. ∎

This result has also been observed in, *e.g.*, [2], [9]. There, it has motivated the use of a dynamic uncertainty overbound in modeling for robust control, although clear guidelines for the selection of such overbound are lacking. Yet, the following proposition confirms that within this coordinate frame, dynamic uncertainty bouding is worthwhile in general.

Proposition 12. Let the robust-control-relevant identification problem in Def. 5 be posed using the dual-Youla uncertainty structure in Prop. 8 with arbitrary coprime factorizations $\{\hat{N}, \hat{D}\}$ and $\{N_c, D_c\}$. Then, selection of the model set complexity $\{n, o\}$ according to (10) using the bound (14) generally yields $o > \mathbf{0}$.

Proof: Consider a nominal model $\hat{P}^{(n)}$ of order n. Let $\gamma^{(n)}(\omega) = \bar{\sigma}(\Delta_o^{(n)}(\omega))$ denote the corresponding model error magnitude. The *smallest* perturbation set required to ensure that $M_o \in \mathcal{M}$ is given by:

$$\mathbf{\Delta}_{\text{min}} := \{\Delta_u \in \mathcal{RH}_\infty \mid \bar{\sigma}(\Delta_u(\omega)) \leq \gamma^{(n)}(\omega)\ \forall\ \omega\}. \quad (16)$$

Dynamic uncertainty modeling requires selection of $\mathcal{W}_\Delta^{(o)}$ such that $\mathbf{\Delta}_{\text{min}} \subset \mathbf{\Delta_u^{(o)}}\ \forall\ o$, see Def. 10. This embedding needs to be tight at ω_{WC} in order to minimize the upper bound (14) on J_{WC}, where:

$$\omega_{\text{WC}} := \arg\max_{\omega} \sup_{\Delta \in \mathbf{\Delta_{min}}} \bar{\sigma}(\hat{M}_{21}(\omega)\Delta(\omega)\hat{M}_{12}(\omega))\,. \quad (17)$$

In general, $\omega_{\text{WC}} \neq \arg\max_{\omega} \gamma(\omega)$. Hence, (10) typically yields an overbound order $o > \mathbf{0}$. ∎

In conclusion, the dynamic overbound should be selected in such a way, that it accounts for the deformation of the uncertainty set $\mathbf{\Delta}_{\text{min}}$ that is caused by $\hat{M}_{21}$ and $\hat{M}_{12}$. Next, it is shown that by judiciously selecting the coprime factorizations that constitute the dual-Youla uncertainty structure, the significance of such deformations is eliminated. This enables a more transparent selection of the uncertainty bound.

B. Robust-control-relevant coprime factors

The coprime factorizations of $\hat{P}$ and C in (11) are nonunique. This may be exploited to create a transparent interconnection between nominal and uncertainty modeling.

Definition 13. [11] Given $\hat{P}$, C and $V = \text{diag}(V_2, V_1)$. Let $\{\tilde{N}_e, \tilde{D}_e\}$ be a left coprime factorization (LCF) of $[CV_2 \ \ V_1]$ with co-inner numerator, where $\tilde{N}_e = [\tilde{N}_{e,2} \ \ \tilde{N}_{e,1}]$. Robust-control-relevant plant coprime factors $\{\hat{N}, \hat{D}\}$ are defined as:

$$\begin{bmatrix}\hat{N}\\ \hat{D}\end{bmatrix} = \begin{bmatrix}\hat{P}\\ I\end{bmatrix}(\tilde{D}_e + \tilde{N}_{e,2}V_2^{-1}\hat{P})^{-1}\,. \quad (18)$$

Definition 14. [11] Given C and $W = \text{diag}(W_y, W_u)$. A (W_y, W_u)-normalized controller coprime factorization $\{N_c, D_c\}$ of C is defined to satisfy:

$$[W_y D_c \ \ -W_u N_c]\,[W_y D_c \ \ -W_u N_c]^H = I. \quad (19)$$

By making use of the above-defined choices for $\{\hat{N}, \hat{D}\}$ and $\{N_c, D_c\}$, it is possible to simplify the bound on the worst-case performance as derived in Lemma 11.

Proposition 15. [11] Using the robust-control-relevant plant coprime factorization in Def. 13 and the (W_y, W_u)-normalized controller coprime factorization in Def. 14, the worst-case performance associated with $\mathcal{P}_{dY}$, see Prop. 8, satisfies the upper bound:

$$J_{\text{WC}}(\mathcal{P}_{dY}^{(n,o)}, C_{\text{exp}}) \ \leq \ J(\hat{P}^{(n)}, C) + \sup_{\Delta_u \in \mathbf{\Delta_u}^{(o)}} \|\Delta_u\|_\infty. \quad (20)$$

The proof in [11] hinges on the fact that $\hat{M}_{21}$ is inner and $\hat{M}_{12}$ co-inner for the specified choice of coprime factorizations. In this case, the general result (14) simplifies, since the $\mathcal{H}_\infty$-norm of a transfer function is invariant for pre-multiplication with an inner matrix and post-multiplication with a co-inner matrix, *cf.* [18, Sect. 13.6].

The importance of Prop. 15 lies in the fact that the worst-case performance associated with a model set is transparently connected to the selected nominal model and uncertainty bound. In contrast to the result in Lemma 11, the size of the admissible set of perturbations directly affects worst-case performance now, *i.e.*, deformations of the perturbation set by $\hat{M}_{21}$ and $\hat{M}_{12}$ no longer play a role. By using this novel uncertainty coordinate frame, important new insights in order selection for robust control are acquired, which provide the main result of this paper: Prop. 16.

Proposition 16. Let the robust-control-relevant identification problem in Def. 5 be posed using the dual-Youla uncertainty structure in Prop. 8, where the robust-control-relevant plant coprime factorization in Def. 13 and the (W_y, W_u)-normalized controller coprime factorization in Def. 14 are chosen. Then, selection of the model set complexity $\{n, o\}$ according to (10) using the bound (20) yields $o = \mathbf{0}$.

Proof: From Def. 10, it follows that for any order o:

$$\sup_{\Delta_u \in \mathbf{\Delta_u}^{(o)}} \|\Delta_u\|_\infty = \|\mathcal{W}_\Delta^{(o)}\|_\infty \geq \max_\omega \gamma(\omega)\,. \qquad (21)$$

The latter bound is tight for the zeroth order overbound $\mathcal{W}_\Delta^{(\mathbf{0})} = \max_\omega \gamma(\omega)$. Hence, (20) simplifies to:

$$J_{\text{WC}}(\mathcal{P}_{dY}^{(n,o)}, C_{\text{exp}}) \leq J(\hat{P}^{(n)}, C) + \max_\omega \gamma(\omega), \qquad (22)$$

which is independent of the overbound order o. Since the order selection criterion of Def. 7 includes the regularization function f_2, the overbound $\mathcal{W}_\Delta^{(\mathbf{0})}$ is selected indeed. ∎

Proposition 16 reveals that dynamic uncertainty modeling does not contribute to minimization of bound (20) on the worst-case performance that is associated with the robust-control-relevant model set in Def. 5. Indeed, intuitively, bringing the *worst-case* performance of the model set closer to the achieved performance $J(P_o, C)$, see (5), requires a fundamental reduction of the *worst* model errors, which can be accomplished through refinement of the nominal model only. Judicious selection of the uncertainty coordinates prevents degradation of worst-case performance without the need for dynamic uncertainty bounding. Nevertheless, there are reasons to consider dynamic uncertainty weights $\mathcal{W}_\Delta^{(o)}$, as discussed in the next section.

VI. Iterative identification and control design

In this section, implications of pursuing an iterative identification and robust control design procedure are investigated. During construction of the model set $\mathcal{P}_{\text{RCR}}$ in Def. 5, a different feedback controller C_{exp} needs to be used than the robust controller that is designed subsequently on the basis of this model set. As a consequence, to evaluate the worst-case performance that is actually achieved, a different closed-loop model set needs to be used than the set defined in Lemma 9.

Lemma 17. [2] Consider an experimental controller $C_{\text{exp}} = N_c D_c^{-1}$. Let $\mathcal{P}_{\text{dY}}(C_{\text{exp}})$ be a robust-control-relevant model set according to Def. 5, which incorporates the dual-Youla uncertainty structure in Prop. 8. Let $C_{\text{RP}}(\mathcal{P}_{\text{dY}}(C_{\text{exp}}))$ be a robust controller that is synthesized on the basis of this model set. The new weighted closed-loop model set $\bar{\mathcal{M}}$, obtained by substitution of $\mathcal{P}_{dY}(C_{\text{exp}})$ and C_{RP} in Fig. 1, is given by:

$$\bar{\mathcal{M}} = \left\{ \bar{M} : w \mapsto z \mid M = \mathcal{F}_u(\bar{M}, \Delta_u),\ \Delta_u \in \mathcal{RH}_\infty \right\} \qquad (23)$$

where $\bar{M} = \left[\begin{array}{c|c} \bar{M}_{11} & \bar{M}_{12} \\ \hline \bar{M}_{21} & \bar{M}_{22} \end{array}\right]$, with:

$$\bar{M}_{11} = -(\hat{D} + C_{\text{RP}}\hat{N})^{-1}(C_{\text{RP}} - C_{\text{exp}})D_c, \qquad (24)$$

$$\bar{M}_{12} = (\hat{D} + C_{\text{RP}}\hat{N})^{-1}\left[C_{\text{RP}}V_1 \quad V_2\right], \qquad (25)$$

$$\bar{M}_{21} = \begin{bmatrix} W_y \\ -W_u C_{\text{RP}} \end{bmatrix} (I + \hat{P}C_{\text{RP}})^{-1}(I + \hat{P}C_{\text{exp}})D_c\,, \qquad (26)$$

$$\bar{M}_{22} = W \begin{bmatrix} \hat{P} \\ I \end{bmatrix} (I + C_{\text{RP}}\hat{P})^{-1}\left[C_{\text{RP}} \quad I\right] V. \qquad (27)$$

It turns out that the order of the uncertainty overbound does have an influence on the worst-case performance achieved by the *newly designed* robust controller.

Proposition 18. Let $\mathcal{P}_{\text{dY}}^{(n,o)}$ be a robust-control-relevant model set that is defined using a dynamic uncertainty bound $\mathcal{W}_\Delta^{(o)}$ of order o, see (9). Let $C_{\text{RP}}(\mathcal{P}_{\text{dY}}^{(n,o)})$ be a robust controller that is synthesized on the basis of this model set. Then, in general, for $o > \mathbf{0}$:

$$J_{\text{WC}}(\mathcal{P}_{\text{dY}}^{(n,o)}, C_{\text{RP}}(\mathcal{P}_{\text{dY}}^{(n,o)})) < J_{\text{WC}}(\mathcal{P}_{\text{dY}}^{(n,\mathbf{0})}, C_{\text{RP}}(\mathcal{P}_{\text{dY}}^{(n,\mathbf{0})})). \qquad (28)$$

Proof: Consider a nominal model $\hat{P}^{(n)}$ of order n. As in (16), let $\mathbf{\Delta}_{\text{min}}$ be the smallest uncertainty set that ensures $M_o \in \mathcal{M}$. In analogy with Def. 6, define the corresponding smallest set of plant models $\mathcal{P}_{\text{min}}^{(n)}$ as:

$$\mathcal{P}_{\text{min}}^{(n)} = \left\{ P \,\middle|\, P = \mathcal{F}_u(\,\hat{H}(\hat{P}^{(n)}),\, \Delta_u)\,,\ \Delta_u \in \mathbf{\Delta}_{\text{min}} \right\}. \qquad (29)$$

Note that actual construction of $\mathcal{P}_{\text{min}}^{(n)}$ demands for an exact description of $\gamma^{(n)}(\omega)$ by the dynamic uncertainty bound $\mathcal{W}_\Delta$, *cf.* Def. 10, which would be of infinite order in general. Since $\mathbf{\Delta}_{\text{min}} \subset \mathbf{\Delta_u^{(o)}}\ \forall\, o$, see Def. 10, $\mathcal{P}_{\text{min}}^{(n)} \subset \mathcal{P}_{\text{dY}}^{(n,o)}$. Hence:

$$J_{\text{WC}}(\mathcal{P}_{\text{min}}^{(n)}, C_{\text{RP}}(\mathcal{P}_{\text{min}}^{(n)})) \leq J_{\text{WC}}(\mathcal{P}_{\text{dY}}^{(n,o)}, C_{\text{RP}}(\mathcal{P}_{\text{dY}}^{(n,o)}))\ \forall\, o. \qquad (30)$$

By virtue of Lemma 17:

$$J_{\text{WC}}(\mathcal{P}_{\text{min}}^{(n)}, C_{\text{RP}}) = \sup_{\Delta_u \in \mathbf{\Delta}_{\text{min}}} \|\mathcal{F}_u(\bar{M}, \Delta_u)\|_\infty \qquad (31)$$

$$\leq J(\hat{P}^{(n)}, C_{\text{RP}}) + \sup_{\Delta_u \in \mathbf{\Delta}_{\text{min}}} \|\bar{M}_{21}\Delta_u(I - \bar{M}_{11}\Delta_u)^{-1}\bar{M}_{12}\|_\infty.$$

The difference between $J(\hat{P}^{(n)}, C_{\text{RP}})$ and $J_{\text{WC}}(\mathcal{P}_{\text{min}}^{(n)}, C_{\text{RP}})$ due to considered uncertainties is largest at:

$$\omega_{\text{RC}} := \arg\max_\omega \sup_{\Delta \in \mathbf{\Delta_{min}}} \bar{\sigma}(\delta(\omega))\,, \quad \text{where} \qquad (32)$$

$$\delta(\omega) := \bar{M}_{21}(\omega)\Delta_u(\omega)(I - \bar{M}_{11}(\omega)\Delta_u(\omega))^{-1}\bar{M}_{12}(\omega)\,.$$

Typically, $\omega_{\mathrm{RC}} \neq \arg\max_\omega \gamma^{(n)}(\omega)$, *cf.* (16). Hence, in general, the bound (30) can be made more tight by constructing an overbound $\mathcal{W}_\Delta^{(o)}$ of order $o > \mathbf{0}$, which yields (28). ■

In Sect. VIII, it is illustrated that indeed, dynamic uncertainty modeling might prove to be advantageous during iterative identification and control design. First, however, the insights that have been obtained in this paper are related to results in set-membership (SM) identification.

VII. INFORMATIVENESS OF THE MODEL SET

An increased confidence in the description of true system dynamics provides opportunities for performance enhancement through robust control, as less uncertainty needs to be accounted for. In fact, since all $M \in \mathcal{M}$ are indistinguishable representations of M_o in robust control design, the *size* of the candidate closed-loop model set $\mathcal{M}$ is a measure of conveyed information, see also [13]. An appropriate measure for the size of $\mathcal{H}_\infty$-norm-bounded model sets is the radius.

Definition 19. [13] Consider a set of *stable* closed-loop models $\mathcal{M} \subset \mathcal{RH}_\infty$. The radius of the set is defined by:

$$r(\mathcal{M}) := \inf_{X \in \mathcal{RH}_\infty} \sup_{M \in \mathcal{M}} \|X - M\|_\infty . \quad (33)$$

The robust-control-relevant model set structure in Sect. V-B establishes a one-to-one connection between the radius of the closed-loop model set and the size of the perturbation set. Importantly, each closed-loop model $\mathcal{F}_u(\hat{M}, \Delta_u)$ is formed by an *affine* mapping in Δ_u, *cf.* (15). This has important geometrical consequences. It is well-known that under an affine mapping, circular sets, like the norm-bounded perturbation set $\mathbf{\Delta_u^{(o)}}$ in Def. 10, undergo a translation, rotation and scaling only. Consequently, $\mathcal{M}$ is circular. Moreover, its center coincides with $\Delta_u = 0$. Hence, the weighted *nominal* feedback loop $\hat{M}_{22}$, see (12), constitutes the (Chebyshev) center of $\mathcal{M}$, which is exploited next.

Proposition 20. Consider the model set in Lemma 9, where the robust-control-relevant plant coprime factorization in Def. 13 and the (W_y, W_u)-normalized controller coprime factorization in Def. 14 are selected. Moreover, perturbations are restricted to the set $\mathbf{\Delta_u^{(o)}}$ in Def. 10. The radius (Def. 19) of this model set is given by:

$$r(\mathcal{M}) = \sup_{M \in \mathcal{M}} \|\hat{M}_{22} - M\|_\infty = \sup_{\Delta_u \in \mathbf{\Delta_u^{(o)}}} \|\hat{M}_{21}\Delta_u \hat{M}_{12}\|_\infty$$
$$= \|\mathcal{W}_\Delta^{(\mathbf{0})}\|_\infty = \max_\omega \gamma(\omega). \quad (34)$$

Proof: The minimum in (33) is attained when the center of the set is substituted for X. Moreover, use is made of the fact that $\hat{M}_{21}$ and $\hat{M}_{12}$ are norm-preserving, *cf.* Prop. 15. Finally, the main result in Prop. 16 is exploited. ■

Remark: whereas conventional uncertainty descriptions like, *e.g.*, the additive perturbation structure [18], provide an affine parametrization of the *open-loop* model set $\mathcal{P}$, see Def. 6, it is the affine parametrization of the *closed-loop* model set $\mathcal{M}$ that enables transparent modeling for robust control.

A. Geometry of the model set

Proposition 20 can be interpreted by bounding J_{WC} in terms of $J(P_o, C)$. Iterative identification and robust control strives to make the gap between the two small (recall (5)).

Proposition 21. Let $\mathcal{P}_{\mathrm{dY}}^{(n,o)}(C_{\mathrm{exp}})$ be a robust-control-relevant model set as considered in Prop. 15. The gap between worst-case performance $J_{\mathrm{WC}}(\mathcal{P}_{\mathrm{dY}}^{(n,o)}, C_{\mathrm{exp}})$ and true system performance $J(P_o, C_{\mathrm{exp}})$ can be expressed in terms of the nominal model-reality mismatch J_{nom}:

$$J_{\mathrm{WC}}(\mathcal{P}_{\mathrm{dY}}^{(n,o)}, C_{\mathrm{exp}}) \leq J(P_o, C_{\mathrm{exp}}) + 2\, J_{\mathrm{nom}}(n), \quad (35)$$
$$\text{where:} \quad J_{\mathrm{nom}}^{(n)} := \|W(T(P_o, C_{\mathrm{exp}}) - T(\hat{P}^{(n)}, C_{\mathrm{exp}}))V\|_\infty .$$

Proof: By making use of Def. 1, it follows from (12) that:

$$J_{\mathrm{nom}}^{(n)} = \max_\omega \bar{\sigma}(M_o(\omega) - \hat{M}_{22}(\omega)) . \quad (36)$$

The weighted true feedback loop M_o is stable. Hence, using (15), $\exists\, \Delta_o \in \mathcal{RH}_\infty$ s.t. $M_o = \hat{M}_{22} + \hat{M}_{21}\Delta_o\hat{M}_{12}$. Since $\hat{M}_{21}$ and $\hat{M}_{12}$ are norm-preserving, *cf.* Prop. 15:

$$\bar{\sigma}(M_o - \hat{M}_{22}) = \bar{\sigma}(\hat{M}_{21}\Delta_o\hat{M}_{12}) = \bar{\sigma}(\Delta_o) := \gamma^{(n)} . \quad (37)$$

As a consequence:
$$\max_\omega \gamma^{(n)}(\omega) = J_{\mathrm{nom}}^{(n)}. \quad (38)$$

The result in (35) is obtained by using (38) together with the triangle inequality:

$$J(\hat{P}^{(n)}, C) \leq J(P_o, C) + J_{\mathrm{nom}}^{(n)} \quad (39)$$
■

From (34) and (38), it is concluded that the radius of the model set is given by the nominal model-reality mismatch, due to usage of the proposed uncertainty coordinate frame. The obtained bound (35) on J_{WC} is understood from Fig. 2, in which the distance between the true (weighted) feedback loop M_o and its worst representation in the model set is depicted for one *single frequency*. Indeed, the nominal model-reality mismatch is encountered twice. (Here, it should be kept in mind that the overbound $\mathcal{W}$ is tight at $\arg\max_\omega \gamma(\omega)$.)

B. Connection to set-membership identification

The work that is presented in this paper connects to SM identification, see, *e.g.*, [4], [8], and [9], although a fundamental difference in philosophy exists. In SM identification, the size of the uncertainty set is determined using a pessimistic (worst-case) interpretation of prior assumptions and available measurement data. An alternative approach is to explicitly quantify the nominal-model reality mismatch on the basis of periodic validation experiments, see, *e.g.*, [16] and [10]. Since this enables an effective elimination of dis-

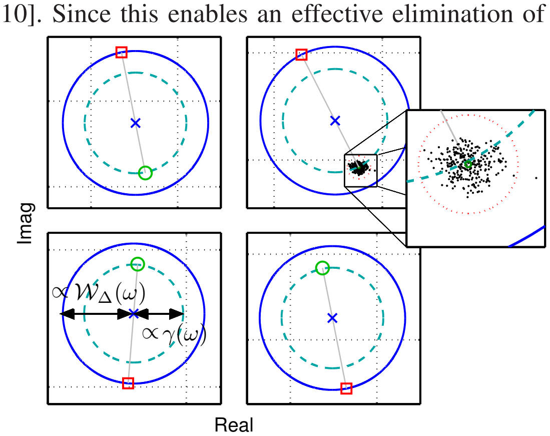

Fig. 2. Single frequency validation plot of $\hat{M}_{22} = J(\hat{P}, C)$ (×), which is extended with the minimum required uncertainty set (dashed) to cover M_o (O). Robust control requires a parametric overbound (solid), which gives rise to the worst-case representation (□) of true feedback behavior in $\mathcal{M}$.

turbance phenomena through averaging, as confirmed in the zoomed are in Fig. 2, an accurate characterization of the minimum uncertainty set needed to explain previously observed system phenomena can be made indeed. In general, this leads to a model set of which the radius is smaller than the *radius of information* needed to cover the pessimistic set of allowed perturbations that is assumed in SM identification. Hence, a more informative description of true system behavior is available, which allows for enhanced robust performance.

VIII. Example

This section illustrates the main result of this paper on an industrial high-precision positioning device, see [6] for details. As motivated in Prop. 21, control-relevant nominal modeling, *cf.* [11], is at the heart of tight modeling for robust control. Table I confirms that the $\mathcal{H}_\infty$ model-reality mismatch J_{nom} decreases monotonically with an increase of the nominal model order n, as is also observed in, *e.g.*, [4].

TABLE I

Model-reality mismatch observed during identification.

Nominal order n	5	7	9	11	13
J_{nom}	2.214	0.168	0.114	0.106	0.105

Next, the 11^{th} order model is validated, see Fig. 3. Note that, since a different linearization of a nonlinear artifact is observed around 200 Hz, $\max_\omega \gamma^{(11)}(\omega) \neq J_{\text{nom}}^{(11)}$, which contradicts (38). Two uncertainty overbounds are shown, which both achieve $J_{\text{WC}} < 3.58$ by virtue of the main result in Prop. 16. Hence, from a robust-control-relevant identification perspective, dynamic overbounding is insignificant.

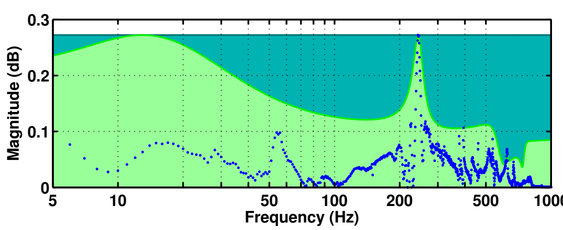

Fig. 3. Validated 11th order model-reality mismatch γ (dotted), with 0^{th} order (light shaded) and 10^{th} order (light shaded) overbound $\mathcal{W}_\Delta$.

Nevertheless, the *dynamic* uncertainty bound establishes a high-fidelity description of resonance phenomena above 600 Hz in the corresponding plant set, see Fig. 4, which allows for explicit compensation in the robust control synthesis step.

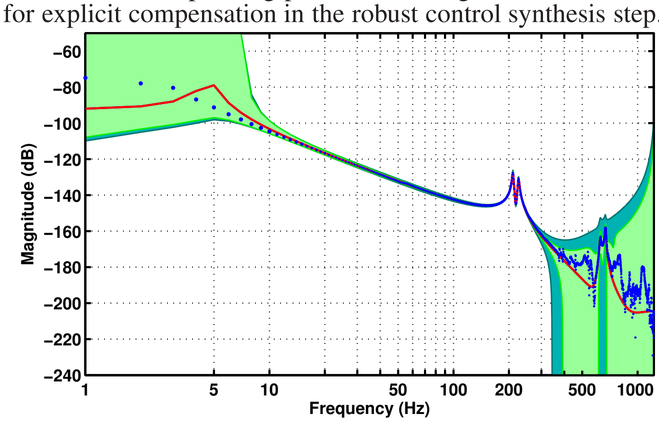

Fig. 4. Identified frequency response function estimate of Po (dotted), 11th order control-relevant nominal model $\hat{P}$ (solid), and the candidate plant set $\mathcal{P}_{dY}$ using the validation-based static uncertainty bound $\mathcal{W}_\Delta^{(\mathbf{0})}$ (dark shaded) and 10th order dynamic uncertainty bound $\mathcal{W}_\Delta^{(10)}$ (light shaded).

IX. Conclusions

In this paper, new insights on model order selection for robust control have been presented. It has been shown that by exploiting freedom in the coprime factorizations that constitute the dual-Youla uncertainty structure, worst-case performance optimization through robust-control-relevant identification becomes invariant for the order of the uncertainty bound. In fact, then, reduction of the radius of the model sets hinges on refinement of the nominal model solely. Through explicit characterization of remaining model errors on the basis of validation experiments, it is possible to obtain a smaller model set radius than the radius of information as obtained by applying set-membership identification techniques, which facilitates enhanced robust performance. Although dynamic overbounding is irrelevant in a single robust-control-relevant identification step, this paper has motivated its potential in iterative identification and robust control design. Current research is focussed on development of paradigms for the selection of dynamic overbounds.

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