# Non-parametric Continuous-time System Identification with Lebesgue-sampled Output Measurements

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## 1 Background

Event-based sampling schemes in control design can lead to improvements in overall performance and resource efficiency [1], with applications in, e.g., network control and incremental encoders [2]. One of the most popular event-based sampling methods is Lebesgue sampling, which consists of sampling the continuous-time signal whenever it crosses fixed thresholds in the amplitude domain.

The problem that is addressed in this work is the estimation of non-parametric continuous-time models from Lebesgue-sampled output data. To this end, we seek an estimator that can 1) provide a *continuous-time* impulse response estimate from possibly noisy and short data records, and 2) exploit the entirety of the information contained in the irregular sampling instants and the bounded intersample behavior.

## 2 Problem formulation

Consider a linear and time-invariant, continuous-time system $G$ (with impulse response $g(t)$), excited by a known input $u(t)$. The noisy output $z(t)$ is Lebesgue-sampled, as in Figure 1. In practice, $z(t)$ is fastly-sampled every $\Delta$ [s] and samples are retrieved only when $z(i\Delta)$ crosses a threshold level. Thus, we have access to $y(i\Delta)=[\eta_i,\eta_i+h]$, where $\eta_i$ is the lower threshold at time $i\Delta$ and $h$ is the threshold amplitude. This setup is presented in Figure 2.

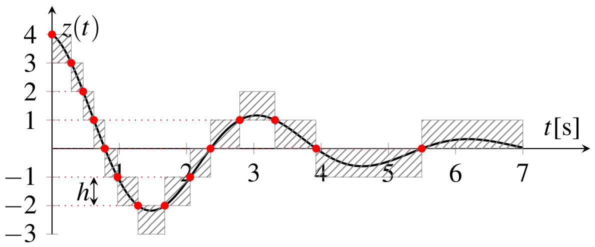


**Figure 1:** Lebesgue sampling of $z(t)$, with threshold amplitude $h=1$.

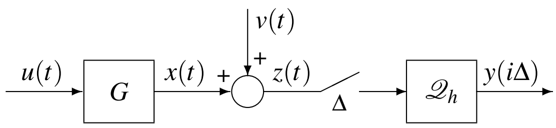


**Figure 2:** Block diagram of the Lebesgue-sampling scheme. The $\mathscr{Q}_h$ block delivers a set-valued signal $y$.

The goal is to derive an identification method that estimates $g(t)$ using $\{u(t)\}_{t\in[0,\Delta N]}$ and $\mathbf{y}_{1:N}=\{y(i\Delta)\}_{i=1}^{N}$.

## 3 Approach

By adopting a kernel-based approach, the goal is to find a regularized estimator of the form

$$\hat{g}=\arg\min_{g\in\mathscr{G}}\left(-\log \mathrm{p}(\mathbf{y}_{1:N}|g)+\gamma\|g\|_{\mathscr{G}}^2\right),$$

where $\mathrm{p}(\mathbf{y}_{1:N}|g)$ is the likelihood function, and $\mathscr{G}$ is a reproducing kernel Hilbert space (RKHS). The impulse response $\hat{g}$ can be obtained via a *finite-dimensional* optimization problem thanks to the Representer Theorem. The finite-dimensional problem that is derived is solved with MAP-EM (MAP Expectation-Maximization), which provides closed-form iterations that are shown to converge to the global optimum if adequately initialized.

Any kernel-based estimator will depend on the choice of hyperparameters. These are computed with Empirical Bayes in this work, from which MAP-EM iterations are also derived. This time the iterations require computing the second moment of a high-dimensional truncated Gaussian. Such matrix can be estimated with Monte Carlo sampling methods.

## 4 Results

We consider $G(s)=1/(0.05s^2+0.2s+1)$. The proposed approach exploits the output intersample knowledge to deliver more accurate models than the standard kernel (Riemann) approach [3], for all values of $h$.

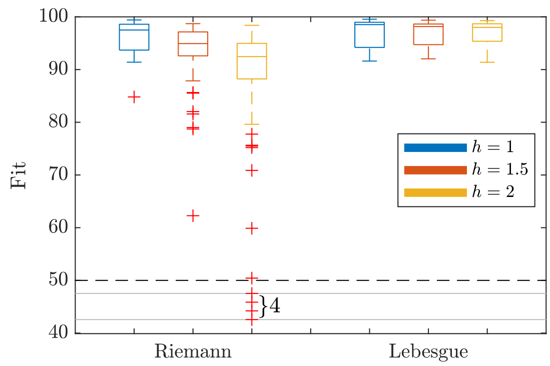


**Figure 3:** Fit boxplots for the standard Riemann approach (left), and the proposed method (Lebesgue, right) for different values of threshold amplitude $h$.

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## References

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[3] G. Pillonetto and G. De Nicolao. "A new kernel-based approach for linear system identification". *Automatica*, 46(1), pp. 81–93, 2010.