Peak Amplitude-Constrained Experiment Design for FRF Identification of MIMO Motion Systems

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Abstract—The accurate identification of Frequency Response Functions (FRF) models is an essential aspect in the design and control of high-precision motion systems. The aim of this paper is to improve FRF identification of multiple inputs multiple outputs motion systems by optimal experiments design under $l_{\infty}$-norm constraints. A two-step experiment design framework is established that exploits a fast smoothing-based algorithm to $l_{\infty}$ signal norm minimization, in conjunction with full multivariable and directional power-constrained excitation design. Experimental results from an industrial active vibration isolation system show a significant improvement of FRF quality compared to traditional design approaches.

Index Terms—Optimal experiment design, system identification, Crest-factor, Frequency response function

I. INTRODUCTION

High-precision motion systems, such as wafer-stages [1], are subject to ever increasing performance demands. These mechatronic systems exhibit complex multivariable dynamics, e.g., due to mechanical resonances, and are typically equipped with a large number of inputs outputs [2]. Meeting the performance requirements demands advanced model-based control strategies [3]. For the majority of these control techniques, the availability of accurately identified non-parametric FRF models [4] is indispensable [5].

The quality of the identified models crucially depends on the excitation signal [6]. Optimal Experiment Design (OED) involves the design of excitations that maximize model quality within experimental limitations. A vast body of literature on OED exists, both for the identification of parametric and non-parametric models, e.g., [4], [7], [8].

Dealing with experimental constraints is a central and necessary aspect in OED to ensure safe identification experiments. The majority of design methods, e.g., [6], [9], [10], consider power-constrained problems, which are typically formulated as convex optimization programs over the input power spectrum. However, in many industrial applications, the constraints are expressed as peak-amplitude bounds on the input and output signals, reflected by the $l_{\infty}$ signal norm. For $l_{\infty}$-norm constraints, power spectrum design approaches cannot directly be used, since the peak-amplitude does not depend on the spectral power only, but also on the phase. The peak-amplitude is a highly nonlinear and non-smooth function of the phases [12], which significantly complicates the design.

To reduce complexity in $l_{\infty}$-norm constrained OED, the problem is typically subdivided into two manageable subproblems: 1) the design of a power spectrum that maximizes model quality subject to power constraints and 2) the design of a signal realization that obeys the prescribed spectrum and achieves low $l_{\infty}$-norm. As such, computational advantages of convex optimization [11] may be exploited in subproblem 1, whereas subproblem 2 reduces to a $l_{\infty}$-norm minimization problem for a given spectrum.

The problem of $l_{\infty}$-norm minimization for prescribed power spectrum is also known as crest-factor minimization, where the crest-factor is the ratio between the peak amplitude and the power [4]. Exact solutions to crest-factor minimization are not known. An overview of existing crest-factor minimization methods for SISO systems is given in [14]. Common techniques are the Schroeder solution [13], and the time-frequency domain swapping method in [15]. Due to the heuristic nature of these techniques, these methods can generally not be extended to multiple inputs multiple outputs (MIMO) systems.

In [12], the crest-factor is optimized by sequentially minimizing $l_p$-norm approximations of the $l_{\infty}$-norm for increasing values of $p$. While applicable to MIMO systems, the method is computationally intensive for large MIMO problems. In prior work [16], a smoothing-based algorithm to crest-factor minimization in MIMO problems is presented that achieves low computational complexity, but the method is not connected to FRF identification.

Although the availability of high-quality FRF models is of essential importance for the control of precision motion systems, fast and accurate FRF identification is hampered by a lack of tractable OED methods that can deal with $l_{\infty}$-norm constraints in large MIMO problems. The aim of this paper is to solve the OED problem for FRF identification of complex multivariable motion systems subject to $l_{\infty}$-norm constraints.

The main contributions of this paper are:

1. A performance analysis of the smoothing-based $l_{\infty}$-norm minimization algorithm (Section III),
2. an embedding of the algorithm in an excitation design framework for optimal FRF identification of multivariable systems (Section IV),
3. an experimental validation on a multivariable Active Vibration Isolation System (Section V).

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II. PROBLEM FORMULATION

A. Frequency Response Function identification

Consider the closed-loop identification setup in Fig. 1. The associated input-output relation in discrete-time for an \( e \)-th identification experiment, is given by

\[
z^e(n) = G(q) * u^e(n) + \nu^e(n),
\]

(1)

Herein, index \( n = [0, \ldots, N-1] \) is the discrete-time index, with sample size \( N \), the measurements are \( z = [y^{T}, u^{T}]^{T} \), and

\[
G(q) = \begin{bmatrix} P(q) \ I \end{bmatrix} S(q), \quad \nu^e = \begin{bmatrix} \nu^e_{y} \ 0 \end{bmatrix} - G(q)K(q)\nu^e_{y},
\]

where \( P \) represents a \( n_y \times n_u \) LTI system to be identified, \( K \) is a stabilizing feedback controller, \( q \) is the forward shift operator, and \( S(q) = (I + K(q)P(q))^{-1} \). Signal \( w \in \mathbb{R}^{n_u} \) is a user-defined excitation signal, and \( y \in \mathbb{R}^{n_y} \) represents the measurements, which are perturbed by an independently and identically distributed random sequence \( \nu^e_{y} \). By collecting the records \( w = [w[1], \ldots, w[n]] \), \( u = [u[1], \ldots, u[n]] \) and \( y = [y[1], \ldots, y[n]] \) of \( n_u \) experiments, the FRF of \( P \) at frequency bin \( k \) is estimated as [4, Ch. 2]:

\[
\hat{P}(k) = Y(k)U^{-1}(k),
\]

(2)

with \( U, Y \) the Discrete Fourier Transform (DFT) of \( u, y \).

B. Optimal experiment design

1) OED problem: The quality of the identified model \( \hat{P} \) in (2) depends on the excitations \( w \). OED consist in the computation of optimal excitation signals \( w^{\text{opt}} \) that satisfy the constraints for all experiments \( e \) and all signals in \( z \), i.e.,

\[
w^{\text{opt}} = \text{minimize} \ J(G(w) \ w)
\]

subject to \( g_i(z^e_i) \leq c_i, \ \forall e, \ i = 1, \ldots, n_z \).

The function \( J \) represents a quality measure of the to-be-identified model, \( g_i \) represent the constraint functions, and \( c_i \) denote the constraint values associated to the \( i \)-th signal in \( z \).

2) Signal peak amplitude constraints: In this paper, signal peak amplitude constraints are considered, \( g_i(z_i) = l_{\infty}(z_i) \), where the \( l_{\infty} \)-norm is defined as follows.

Definition 1: The \( l_{\infty} \)-norm of a scalar-valued signal \( x(n) \) is defined as its absolute peak value in the interval \([0, N-1]\),

\[
l_{\infty}(x) = \max_{n \in [0, N-1]} |x(n)|.
\]

(4)

3) Signal parametrization: Band-limited multisine signals \( w \) are considered, i.e., for the \( j \)-th input and the \( e \)-th experiment,

\[
w^e_{j}(n, A, \Phi) = \sum_{k=1}^{N_k} a^e_{jk} \cos \left( \frac{2 \pi k n}{N} + \phi^e_{jk} \right),
\]

(5)

where \( N_k \leq 1/2N-1 \) is the user-defined excitation frequency band, and \( a_{jk} > 0 \ \forall j, k \) and phases \( \phi_{jk} = [0, 2\pi], \ \forall j, k \) are collected in \( A \) and \( \Phi \), respectively. The output signals \( z^e_{i} \) in (1) for inputs \( w^e \) in (5), assuming \( \nu^e_{e} = 0 \), are given by

\[
z^e_{i}(n) = \sum_{j=1}^{n_u} \sum_{k=1}^{N_k} a^e_{ijk} |G_{ijk}| \cos \left( \frac{2 \pi k n}{N} + \phi^e_{jk} + \angle G_{ijk} \right),
\]

(6)

where \( |G_{ijk}| \) and \( \angle G_{ijk} \) denote the magnitude and the phase of entry \([i, j]\) of \( G \) at the \( k \)-th spectral line. The experiment index \( e \) is omitted in the following, unless otherwise noted.

C. Optimization problem aspects

Solving the OED problem (3) is a complex problem in general, both from a fundamental and computational perspective, especially for MIMO systems. The complexity is due to:

a) the constraint function \( l_{\infty}(z_i) \) is non-convex in the variables \( \Phi \), hence local minima should be dealt with,

b) the \( l_{\infty} \)-norm is a non-smooth and not everywhere differentiable function, which prohibits employing standard optimization techniques,

c) the number of design variables in \( A, \Phi \) is typically large, especially for large MIMO systems,

d) For MIMO systems, the problem consists in the design of both the excitation magnitudes and the directions, the latter of which are not present for SISO systems [17].

D. Two-step design approach

To obtain a tractable computational method to solve (3), the problem is separated into two subproblems (SP):

SP1: \( l_{\infty} \)-norm minimization over the variables \( \Phi \), for prescribed magnitudes \( A \),

SP2: computation of spectral magnitudes \( A \) that minimize \( J \), subject to power-constraints.

Such separation is common, as evidenced by, e.g., [12]–[15] that aim at solving SP1, whereas [6], [9], [10], [17] address SP2. In this paper, a framework is established wherein the two subproblems are addressed in view of the joint goal of FRF identification of MIMO systems. The interplay between the two subproblems plays a particularly important role in MIMO problems, due to directionality in multivariable excitations.

In Section III, SP1 is addressed using the algorithm presented in prior work [16], which is particularly suitable for large MIMO problems. Herein, items a) - c) are explicitly addressed. SP2 is solved in Section IV by embedding the algorithm in a multivariable and directional OED framework for FRF identification, hereby addressing item d).

III. SMOOTHING-BASED ALGORITHM TO \( l_{\infty} \)-NORM MINIMIZATION

In this section, SP1 defined in Section II-D is solved by the smoothing-based algorithm presented in prior work [16]. The key idea of smoothing is that it enables first or second order optimization procedures, while reducing the susceptibility to local minima, hence jointly addressing items a) and b) in Section II-C. These capabilities are illustrated in this section, and the algorithm performance is compared to existing methods.
A. Minimization objective

By using the equivalence between the constraint set \( l_\infty(z_i) \leq c_i, \forall i \) in (3) and the constraint \( l_\infty(\bar{z}) \leq 1 \) with \( \bar{z} = [z_1/c_1, \ldots, z_n/c_n] \), the subproblem SP1 is formulated as the unconstrained \( l_\infty \)-norm minimization problem

\[
\minimize_{\Phi} l_\infty(\bar{z}(A, \Phi, G)).
\]  

(7)

Herein, spectral magnitudes \( A \) are assumed given from the solution of SP2, whereas prior system knowledge of \( G \) required to solve (7) is assumed available, e.g., from preliminary identification experiments [18]. The dependency upon \( A \) and \( G \) is omitted in the notation in the following. The objective function (7) is non-convex and non-smooth in the parameters \( \Phi \). This is visualized in the following example.

Example 1: Consider \( w \) in (5) with \( N_k = 51, a_k = \sqrt{2/N_k}, \forall k \) and random phases \( \phi_k \in [0, 2\pi), k = 1, \ldots, N_k - 1 \). Fig. 2 shows the function \( l_\infty(w(\Phi)) \). The function is non-smooth, e.g., around the minima, and evidently non-convex. An approach that addresses these aspects is presented next.

B. Smoothing function

To tackle the non-smoothness of the constraints in (7), the \( l_\infty(\bar{z}) \) function is approximated by the following smooth function, which is based on the exponential barrier function presented in [19], [20] for solving inequality-constrained and ill-conditioned non-linear programs.

Lemma 1: Consider the function

\[
L(\bar{z}, \sigma) = \sigma \ln \sum_{n=0}^{N-1} \exp \left( \frac{\bar{z}^2(n)}{\sigma} \right)
\]

(8)

for \( \sigma > 0 \). Then \( \lim_{\sigma \to 0} L(\bar{z}, \sigma) = l_\infty^2(\bar{z}) \) and \( L(\bar{z}, \sigma) \) is an everywhere twice differentiable function.

A proof is found in [21], [22]. The smoothing mechanism is illustrated in the following example.

Example 2: Consider \( w \) as in Example 1. Fig. 2 shows \( L_{1/2}(w, \sigma) \) for \( \sigma = \{1, 0.8, 0.6, 0.4, 0.2\} \) as function of \( \phi_{51} \). The smoothness and differentiability of \( L(\bar{z}, \sigma) \) is exploited in a gradient-based optimization approach in the next section.

C. Gradient-based algorithm

The smooth approximation function (8) is exploited in an algorithm for accurately solving (7), by a joint minimization of the function \( L(\bar{z}, \sigma) \) and a gradual decrease of the smoothing parameter \( \sigma \). Since excitation design problems tend to be large in terms of \( N \) and \( N_k \), the algorithm is particularly aimed at achieving low computational complexity. This is realized by employing the Polak-Ribiere Conjugate Gradient (PRCG) method [23], in conjunction with a fast method for the computation of the gradient \( \nabla_{\Phi} L \) presented in [16].

Consider Algorithm 1. In step 1, the initial search direction is the negative gradient direction. The notation \( L^i = L(\bar{z}(\Phi^i), \sigma^i) \) is used. In step 2, to achieve guaranteed descent of \( L(\bar{z}(\Phi^i), \sigma^i) \) for fixed \( \sigma^i \), the step size \( \alpha^i \) is selected to satisfy the modified strong Wolfe line search conditions in [24]. In step 3 the \( \Phi \)-parameters are updated. In step 4, the smoothing parameter \( \sigma \) is controlled by the rate of descent of \( L \). In step 5, a new search direction \( d^{i+1} \) is generated. Specifically, after a reduction of \( \sigma \), parameter \( q \to 1 \), and (9) reduces to a gradient descent direction. Otherwise, \( q = 0 \), (9) is a PRCG direction [23].

The sequence \( \{L^i\} \) generated by Algorithm 1 is monotonically decreasing and \( \Phi^i \) attains a stationary point of \( l_\infty(\bar{z}(\Phi)) \) for \( i \to \infty \), which can be proven along the lines of the proof of [16, Theorem 1].

The benefits of the gradual reduction of the smoothing parameter \( \sigma \) in step 5 are demonstrated in the next section.

D. Avoiding local minima: a Monte Carlo analysis

A key aspect of Algorithm 1 is that a gradual decrease of the \( \sigma \) parameter reduces the susceptibility to local minima. The reason is that for \( \sigma \to \infty \) certain local minima vanish, as is formalized in [16]. This phenomenon is also observed in Fig. 2, and is further investigated in the following example.

Example 3: Consider a multisine \( w \) with \( N_k = 52, a_k = \sqrt{2/N_k}, \forall k \) and random phases \( \phi_k \in [0, 2\pi), k = 1, \ldots, N_k - 2 \). Algorithm 1 is executed for 100 initial conditions of \( \{\phi_{51}, \phi_{52}\} \), distributed over the grid as shown in Fig 3(a). The percentage of solutions that attains the global minimum for different values of the initial smoothing param-
parameter $\sigma^0$ is shown in Fig. 3(b). Evidently, for small values of $\sigma^0$, locally optimal solutions are obtained, whereas choosing larger values of $\sigma^0$ enables avoiding local minima entirely. This illustrates the strength of the gradual smoothing-based approach in Algorithm 1.

### E. A performance benchmark

The performance of Algorithm 1 is compared to existing techniques for crest-factor minimization, for a scalar-valued multisine $w$ in (5) of length $N = 10^4$, with $N_k = 4999$ harmonics. Furthermore, $\phi_k = \sqrt{2/N_k}$, $\forall k$ and the initial phases are randomly selected from $\phi_k \in [0, 2\pi)$, $\forall k$. The results are shown in Fig. 4. The Schroeder heuristic [13] gives $l_{\infty}(w) = 1.46$. The random method generates random realizations of $w$ and retains the realization with the lowest peak-amplitude. This method yields $l_{\infty}(w) > 3$ in 70s. The time-frequency domain swapping algorithm [15] yields $l_{\infty}(w) = 1.44$ in 0.6s computation time. The $l_p$-norm algorithm [12] for $p = (2^3, 2^6, 2^8, 2^{10})$ achieves a substantially lower amplitude of $l_{\infty}(w) = 1.16$, but requires a computation time of 70s. The smoothing algorithm (Algorithm 1), achieves the lowest peak with $l_{\infty}(w) = 1.13$ and is $\sim 80$ times faster than the $l_p$ algorithm, which is due to the efficient computation of the gradient in [16], combined with the first order PRCG optimization strategy.

Besides the excellent performance of the smoothing algorithm, an additional major advantage is its suitability for MIMO problems. This is exploited for the design of multivariable excitations in the next section.

### IV. EMBEDDING IN MULTIVARIABLE OED FRAMEWORK

The optimization-based approach, in conjunction with the fast computational method, make the smoothing-based algorithm particularly suitable to solving large MIMO identification problems. In this section, the algorithm is embedded in an OED framework for accurate FRF identification of MIMO systems, which enables solving SP2.

#### A. Multivariable OED under element-wise power constraints

SP2 defined in Section II-D involves the computation of multivariable excitations that minimize a measure of model uncertainty, subject to power constraints.

1) **Element-wise constraints:** The excitations determined in SP2 form the basis for the subsequent design under $l_{\infty}$-norm constraints in SP1. To achieve low conservatism when connecting the two design steps, it is crucial that the constraints selected in SP2 are compatible with the $l_{\infty}$-norm in the subsequent design step. In particular, this requires adopting the element-wise property of the constraints in (3), i.e., $g_i$ is imposed on the individual elements $z_i^{[e]}$, $\forall i, e$. Therefore, the considered power constraints are of the form $g_i(z_i) = l_2(z_i^{[e]}) \leq p_i$, $\forall i, e$, with $p_i$ the constraint values and where the $l_2$-norm is defined as follows.

**Definition 2:** The $l_2$-norm of the scalar-valued signal $x(n)$ in the interval $[0, N - 1]$ is defined as

$$l_2(x) = \left( \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2 \right)^{\frac{1}{2}}. \quad (10)$$

2) **OED problem formulation:** The power-constrained OED problem is most naturally addressed in the frequency-domain. The considered program is formulated as

\[
\text{minimize } \mathcal{J}(W) \\
\text{subject to } \sum_{k=1}^{N_k} \text{Tr} \left( H_i(k) W^{[e]}(k) W^{[e]H}(k) \right) \leq p_i \quad \forall i, e. 
\]

(11)

Herein, $W^{[e]}$ is the DFT of $w^{[e]}$, the constraints represent the element-wise power constraints (via Parseval’s theorem), and $H_i(k) = G_i^H(k) G_i(k)$. Furthermore, the objective function $\mathcal{J}$ in (11) represents the cumulative variance in the FRF model,

\[
\mathcal{J}(W) = \sum_{k=1}^{N_k} \text{Tr} C_{p}(k), 
\]

(12)
where $C_p(k)$ is the covariance matrix given by [4], [17],
\[
C_p(k) = \left( S(k) \sum_{e=1}^{n_u} W[k](k)W[k]^H(k)S^H(k) \right)^{-1} \otimes C_Y(k).
\]
with $C_Y(k)$ the covariance associated to $Y$, see [4, Ch. 2].

3) Solving the OED problem: Program (11) is non-convex for MIMO systems. A fast and accurate method to solving this problem approximately is the Relaxation and Randomization algorithm [17], which uses a convex relaxation combined with a stochastic optimization step.

B. Embedding the $l_\infty$-norm optimization step

For MIMO systems, the optimal excitations to (11), say $\hat{W}[e]$, are multivariable, i.e., they have a frequency-wise directionality that depends on the directionality of the system [17]. These directions are determined by both the excitation magnitudes and the phases. The latter implies that the phases, at least to a certain degree, are predetermined in the solution to (11). To guarantee that the performance achieved by $\hat{W}$ in SP2 is maintained in the subsequent $l_\infty$-norm optimization step in SP1, the phase design in SP1 must be direction-preserving. This is achieved by exploiting the non-uniqueness in the solution to (11), i.e.,
\[
\hat{W}[e](k)\hat{W}[e]^H(k) = \hat{W}[\epsilon](k)e^{j\phi_k} (e^{j\phi_k})^H \hat{W}[\epsilon]^H(k).
\]
So, the remaining design freedom per experiment is in the selection of the frequency-wise phase parameter $\phi_k^\epsilon$. Under preservation of the excitation directions of $\hat{W}$, the outputs of the MIMO system $G$ in (1) are given by
\[
z_i^\epsilon(n) = \sum_{k=1}^{N_k} |M_k^\epsilon| \cos \left( \frac{2\pi kn}{N} + \phi_k^\epsilon + \angle M_k^\epsilon \right),
\]
where $M_k^\epsilon = G_{ik} \hat{W}[\epsilon](k)$ is a mapping in which the solution $\hat{W}[\epsilon]$ to SP2 is absorbed. Algorithm 1 is readily applicable to solve the associated $l_\infty$-norm minimization problem. The resulting two-step OED framework is experimentally validated in the next section.

V. EXPERIMENTAL VALIDATION

In this section, the presented framework is experimentally validated on an Active Vibration Isolation System (AVIS), and the performance is compared to a traditional design approach.

A. Active Vibration Isolation System

The AVIS, depicted in Fig. 5, has six inputs and six outputs. Its six rigid body motion degrees of freedom motion are closed-loop controlled using a PID controller that achieves a bandwidth of 20Hz. Due to electro-mechanical limitations, peak amplitude constraints are imposed onto all six input and six output signals. Appropriate scaling is applied such that all constraint values are equal to one, $c_i = 1, \forall i$.

B. Experiment design

Multisine excitation signals are considered with sample size $N = 5000$ and $N_k = 1000$ harmonics. For comparison purposes, two sets of experiments are performed:

1) Optimal SIMO excitation design: This design uses a traditional excitation approach, in which a single input is excited in each of the $n_u$ experiments. The excitation magnitudes are computed by the convex power-constrained optimization method [25], and the phases by Algorithm 1.

2) Optimal MIMO excitation design: In these experiments, the plurality of the inputs is exploited using multivariable and directional excitation signals, which are computed by the approach presented in this paper.

C. Results

1) Computational performance: The SIMO design requires a computation time of 2.1s, where 0.3s is consumed by the power-constrained optimization (SP2) and 1.8s by the peak-constrained optimization (SP1). The MIMO design requires a time of 9.1s, with 3.5s and 5.6s for the respective optimizations. This illustrates the good computational performance of the smoothing based approach for MIMO problems.

2) Computed signals: The output signals $y$ computed in the SIMO and MIMO designs are shown in Fig. 6. The MIMO approach delivers signals that tightly satisfy the $l_\infty$-norm constraints for each of the elements $i$, i.e. A similar observation is made for the input signals $u$ (not depicted). Thus, in this approach, the available design freedom is well-exploited, which is beneficial for achieving a good signal-to-noise ratio. This is in sharp contrast to the SIMO approach, by which the constraints are tight only in the diagonal elements.

3) Frequency Response Function: A 2x2 selection of the entries of the identified FRF models is shown in Fig. 7. A significant reduction of the uncertainty in observed when using the MIMO design, especially below 120Hz. This is supported by a reduction of a factor 11 of the cost $J$ using the MIMO design, compared to the SIMO design. Concludingly, these results verify the presented multivariable OED approach and demonstrate that these techniques enable a quality improvement of FRF models.

![Fig. 5. AVIS experimental setup.](image-url)
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