

# Flexible ILC: Towards a Convex Approach for Non-Causal Rational Basis Functions

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**Abstract:** Iterative learning control (ILC) is subject to a trade-off between effective compensation of repeating disturbances, and amplification of non-repeating disturbances. Although important progress has been made in enhancing the flexibility of ILC to non-repeating tasks by means of basis functions, at present high performance comes at the cost of non-convex optimization. The aim of this paper is to develop a convex approach to ILC with rational basis functions. A key aspect of the proposed approach is the use of orthonormal basis functions in  $\mathcal{L}_2$ , such that non-causal control actions can be utilized. The benefits of using non-causal rational basis functions in ILC are demonstrated by means of a relevant example.

*Keywords:* Iterative learning control; preview control; rational basis functions; motion control

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## 1. INTRODUCTION

Iterative learning control (ILC) (Bristow et al., 2006) is subject to a trade-off: disturbances that repeat each iteration can be effectively compensated, while non-repeating disturbances are typically amplified (Mishra and Tomizuka, 2009; Gao and Mishra, 2014; Bolder et al., 2015). The key idea behind ILC is to iteratively improve the control action by learning from previous iterations, or executions, of the same task. The effectiveness for repeating disturbances can to a large extent be attributed to non-causality in the time domain, see, e.g., Goldsmith (2002), which allows to anticipate future repeating disturbances. This is a crucial difference with feedback control, which is inherently causal. Yet, ILC is strictly causal in the iteration domain, which is the key reason that ILC typically amplifies non-repeating disturbances.

This potentially harmful effect of learning control has led to a significant research effort to enhance the flexibility of ILC to non-repeating tasks. Hoelzle et al. (2014) construct tasks from a set of standardized subtasks, providing flexibility for a restricted set of tasks. Instead of learning subtasks, other approaches parametrize the feedforward in terms of basis functions. In Phan and Frueh (1996) and Van de Wijdeven and Bosgra (2010), a linear combination of polynomial basis functions is used. Importantly, this yields a convex optimization problem. However, the performance is limited due to the polynomial nature of the feedforward controller. For high performance, the controller should approximate the inverse system (Devasia, 2002). Especially for lightly damped systems, this requires the use of very high order polynomials, which are susceptible to noise acting on the system.

A rational combination of basis functions is used in Bolder et al. (2015), Van Zundert et al. (2016a), Blanken et al. (2016), which has shown to significantly improve perfor-

mance. However, in sharp contrast to the traditional linear parametrization, the optimization problem is non-convex, since also the poles are optimized. As a consequence, nonlinear optimization algorithms are required, for which (global) convergence cannot be guaranteed in general.

Although important progress has been made in enhancing the flexibility of ILC to non-repeating tasks, at present improved performance comes at the expense of non-convex optimization and consequences thereof. The aim of this paper is to develop a new approach that bridges the gap between earlier approaches, and combines their merits: i) convex optimization, and ii) a rational feedforward controller. The approach employs a linear combination of rational basis functions. Indeed, this preserves the analytic solution of Phan and Frueh (1996) and Van de Wijdeven and Bosgra (2010), while fixing the poles of basis functions enables high performance for general rational systems.

The proposed methodology builds on the use of classical orthonormal rational basis functions (Takenaka, 1925; Malmquist, 1925; Bultheel et al., 1999). These have received significant attention in system identification, see, e.g., Ninness and Gustafsson (1997), Ninness et al. (1999), Akçay (2000), and Heuberger et al. (2005). Interestingly, the bases are typically used to parametrize causal models, hence spanning  $\mathcal{H}_2$ . In sharp contrast, in the present paper, basis functions that span  $\mathcal{L}_2$  are developed and their use in ILC is advocated. Particularly, the proposed approach enables infinite pre-actuation and post-actuation through stable inversion (Zou and Devasia, 1999; Sogo, 2010). In addition, by constructing a basis that spans  $\mathcal{L}_2$ , ILC can achieve optimal performance for any system in  $\mathcal{L}_2$ .

The main contribution of this paper is an optimization-based framework for ILC with non-causal rational basis functions, which has an analytic solution. The approach combines the benefits of the approaches in Phan and Frueh

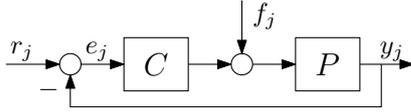


Fig. 1. Closed-loop control configuration.

(1996) and Van de Wijdeven and Bosgra (2010) on the one hand, and Bolder et al. (2015), Van Zundert et al. (2016a), and Blanken et al. (2016) on the other hand. The benefits of the proposed approach are demonstrated through simulations.

The outline of this paper is as follows. In Section 2 the control problem is formulated. In Section 3, the proposed approach using non-causal rational basis functions is presented, and connections with pre-existing approaches are provided. An example in Section 4 confirms the benefits of the proposed approach. Conclusions are given in Section 5.

*Notation:* All systems are discrete-time, single-input single-output (SISO), and linear time-invariant. The complex indeterminate  $z \in \mathbb{C}$  is omitted when this does not lead to any confusion. The following standard notation is used, see, e.g., Heuberger et al. (2005). Let  $\mathbb{D}$  denote the open unit disk:  $\{z, |z| < 1\}$ ,  $\mathbb{E}$  the complement of the closed unit disk:  $\{z, |z| > 1\}$ , and  $\mathbb{T}$  the unit circle:  $\{z, |z| = 1\}$ .  $\mathcal{L}_2$  denotes the set of complex functions that are square integrable on  $\mathbb{T}$ , and the real rational subspace of  $\mathcal{L}_2$  is denoted  $\mathcal{RL}_2$ .  $\mathcal{H}_2$  denotes the set of complex functions that are square integrable on  $\mathbb{T}$  and analytic in  $\mathbb{E}$ .  $\mathbb{R}[z^{-1}]$  denotes the polynomial ring in indeterminate  $z^{-1}$  with coefficients in  $\mathbb{R}$ , and  $\mathbb{R}[z, z^{-1}]$  denotes the Laurent polynomial ring in indeterminate  $z$  with coefficients in  $\mathbb{R}$ . Contrary to regular polynomials, Laurent polynomials include both positive and negative exponents of the indeterminate. Signals are often assumed to be of length  $N$ . For a vector  $x \in \mathbb{R}^N$ , the weighted two-norm is given by  $\|x\|_W^2 = x^T W x$  with  $W \in \mathbb{R}^{N \times N}$ . The  $i$ -th element of  $x$  is denoted  $x[i]$ .  $W$  is positive definite if  $x^T W x > 0, \forall x \neq 0$ .

## 2. PROBLEM FORMULATION

### 2.1 System Description

Consider the control configuration in Figure 1. The system  $P(z)$  is assumed to be given by the rational representation

$$P(z) = \frac{B_0(z)}{A_0(z)},$$

with  $A_0(z), B_0(z) \in \mathbb{R}[z^{-1}]$ . The control configuration consists of a stabilizing feedback controller  $C(z)$ , and a feedforward controller  $F(z)$ . A sequence of finite time tasks is performed, denoted by index  $j = 0, 1, 2, \dots$ . Furthermore,  $r_j$  denotes the reference,  $y_j$  the output signal, and  $f_j$  the feedforward signal. The error signals  $e_j$  and  $e_{j+1}$  in iterations  $j$  and  $j+1$  are given by

$$e_j = S r_j - S P f_j, \quad (1)$$

$$e_{j+1} = S r_{j+1} - S P f_{j+1}, \quad (2)$$

with sensitivity  $S = (I + PC)^{-1}$ .

### 2.2 Norm-Optimal ILC

Iterative learning control can significantly improve the performance of systems that perform repeating tasks, i.e.,

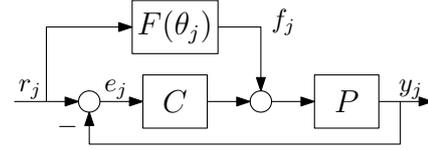


Fig. 2. Control configuration for non-repeating tasks: ILC with parametrized feedforward filter  $F(\theta_j)$ .

$r_{j+1} = r_j, \forall j$ . The goal is to minimize the error signal by updating the feedforward using measured data from previous tasks. Eliminating  $S r_{j+1} = S r_j$  in (2) using (1) yields the error propagation from iteration  $j$  to  $j+1$ :

$$e_{j+1} = e_j + S P (f_j - f_{j+1}). \quad (3)$$

In norm-optimal ILC, see, e.g., Gunnarsson and Norrlöf (2001) and Bristow et al. (2006), the feedforward signal  $f_{j+1}$  for iteration  $j+1$  is determined as

$$f_{j+1} = \arg \min_{f_{j+1}} V(f_{j+1}),$$

where the performance criterion  $V(f_{j+1})$  is defined next.

**Definition 1** (Performance criterion norm-optimal ILC). *The performance criterion for norm-optimal ILC is given by*

$$V(f_{j+1}) = \|e_{j+1}(f_{j+1})\|_{W_e} + \|f_{j+1}\|_{W_f} + \|f_{j+1} - f_j\|_{W_{\Delta f}}, \quad (4)$$

where  $W_e \succ 0, W_f, W_{\Delta f} \succeq 0$  are user-defined positive-(semi)definite weighting matrices, and  $e_{j+1}$  is given by (3).

In view of (3), norm-optimal ILC computes a signal  $f_{j+1}$  that is optimal for (4) for a specific task  $r_{j+1} = r_j$ . However, when the key assumption underlying ILC is violated, i.e.,  $r_{j+1} \neq r_j$ , the performance may severely deteriorate, see, e.g., Mishra and Tomizuka (2009), Hoelzle et al. (2014), and Blanken et al. (2016). This is the key motivation to introduce basis functions in ILC.

### 2.3 Problem Description and Outline

The flexibility of ILC to non-repeating tasks is enhanced by parametrizing  $f_{j+1}$  in terms of a filter and  $r_{j+1}$  as

$$f_{j+1} = F(\theta_{j+1}) r_{j+1}, \quad (5)$$

with parameters  $\theta_{j+1} \in \mathbb{R}^{n_\theta}$ , see Figure 2. In sharp contrast with standard norm-optimal ILC,  $f_{j+1}$  is an explicit function of  $r_{j+1}$ . Substitution of (5) into (2) yields

$$e_{j+1} = S(I - P F(\theta_{j+1})) r_{j+1},$$

which shows that perfect tracking, i.e.,  $e_{j+1} = 0$ , is achieved if  $F(\theta_{j+1}) = P^{-1}$ , irrespective of  $r_{j+1}$ . This requires an appropriate selection of  $F(\theta_{j+1})$ .

The contribution of this paper is a parametrization of  $F(\theta)$  in terms of basis functions that allows

- i) a convex optimization problem;
- ii) inversion of general systems  $P$  with good convergence rate, i.e., a small number of parameters required;
- iii) and infinite pre-actuation and post-actuation.

In the next section, the proposed approach to ILC with basis functions is presented, which is the main contribution of this paper. Consequently, connections to pre-existing approaches using basis functions are established. In Section 4, the benefits of the proposed approach compared to existing approaches are demonstrated through an example.

### 3. A CONVEX APPROACH TO FLEXIBLE ILC WITH NON-CAUSAL RATIONAL BASIS FUNCTIONS

In this section, a convex approach to ILC is proposed that uses rational basis functions, i.e., with prespecified poles. The proposed approach combines the advantages of pre-existing approaches: i) a convex optimization problem, ii) inversion of general rational systems, iii) and infinite pre-actuation and post-actuation. The constructed basis functions are orthonormal, which potentially enhances numerical properties of the optimization problem (Ninness et al., 1999). The following parametrization of  $F(\theta)$  is used.

**Definition 2** (Linear feedforward parametrization).

The feedforward controller  $F(\theta)$  parameterized in terms of a linear basis  $\mathcal{F}_{lin}$  is given by

$$\mathcal{F}_{lin} = \{A(\theta) \mid \theta \in \mathbb{R}^{n_\theta}\}$$

with

$$A(z, \theta) = \theta^\top \Psi(z) = \sum_{i=1}^{n_\theta} \theta[i] \psi_i(z),$$

where  $\psi_i(z)$ ,  $i = 1, \dots, n_\theta$  are basis functions, and  $\Psi(z) = [\psi_1(z), \dots, \psi_{n_\theta}(z)]^\top$ .

Here, the basis functions are designed as non-causal rational orthonormal basis functions (ROBF)  $\psi_i \in \mathcal{RL}_2$ . The orthonormality is with respect to the inner product on  $\mathcal{L}_2$ :

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \psi_k(e^{i\omega}) \psi_l(e^{i\omega}) d\omega = \begin{cases} 1 & \text{if } k = l, \\ 0 & \text{if } k \neq l. \end{cases}$$

In Subsection 3.1, the advantages are presented of selecting rational basis functions in  $\mathcal{L}_2$ . Next in Subsection 3.2, a general solution for ILC with ROBF is proposed. Subsequently, the approach for the construction of non-causal ROBF is presented in Subsection 3.3. Finally, connections to pre-existing approaches are provided in Subsection 3.4.

#### 3.1 Non-Causal Feedforward Through $\mathcal{L}_2$ -Basis Functions

The motivation for the use of ROBF is twofold. First, the benefits of rational functions are discussed. Second, the use for orthonormal functions is motivated.

The key aspect in the design of rational basis functions is specifying the poles. Indeed, the Laurent polynomials used in existing ILC approaches with basis functions have poles at  $z = 0$ , see, e.g., Van de Wijdeven and Bosgra (2010). As such, many parameters are required to accurately model systems with slow poles. Furthermore, the amount of preview is determined by the order of the used Laurent polynomials, which is finite in practice. This motivates to use general rational functions, and exploit the freedom in choosing the pole locations. Interestingly, poles in  $\mathbb{E}$  can be used to generate infinite pre-actuation, and poles in  $\mathbb{D}$  enable infinite post-actuation. This is explained by interpreting  $F(\theta)$  as a bounded and non-causal operator on  $\mathcal{L}_2$ , see, e.g., Vinnicombe (2001, Section 1.5), rather than an unbounded and causal operator on  $\mathcal{H}_2$ . This duality of interpretation has interesting consequences for rational feedforward. Using stable inversion, poles in  $\mathbb{E}$  can be used to generate non-causal feedforward. Decompose  $f_{j+1}$  as

$$f_{j+1} = F_s(z, \theta_{j+1}) r_{j+1} + F_u(z, \theta_{j+1}) r_{j+1}, \quad (6)$$

where  $F_s$  and  $F_u$  contain the stable and unstable dynamics, respectively. Then, the unstable dynamics  $F_u$  are

filtered in backward time with suitable boundary conditions to obtain non-causal feedforward actions, i.e., infinite preview. Detailed state-space expressions for (6) are given in, e.g., Sogo (2010) and Van Zundert et al. (2016b).

The main motivation for using *orthonormal* rational basis functions is improved numerical properties compared to general rational functions (Ninness et al., 1999). Interestingly, the orthonormality does not influence the potential performance of ILC: any orthonormal basis is equivalent under a linear transformation to any other non-orthonormal basis with the same prespecified poles.

Before a construction procedure for non-causal ROBF is given in Subsection 3.3, a general solution for ILC with, possibly orthonormal, rational basis functions is presented.

#### 3.2 ILC with Rational Basis Functions

In this subsection, a solution is presented for ILC with general rational basis functions. The approach assumes feedforward parametrization  $\mathcal{F}_{lin}$ , see Definition 2.

The optimal parameter update is given by

$$\theta_{j+1} = \arg \min_{\theta_{j+1}} V(\theta_{j+1}), \quad (7)$$

with  $V(\theta_{j+1})$  as in (4). Since (4) is quadratic in  $\theta_{j+1}$ , an analytic solutions exists to (7). Given  $SP$  and basis functions  $\Psi$ , the solution is given by the general form

$$\theta_{j+1} = Q\theta_j + Le_j, \quad (8)$$

where

$$\begin{aligned} Q &= \left[ \Psi_{r_j}^\top ((SP)^\top W_e SP + W_f + W_{\Delta f}) \Psi_{r_j} \right]^{-1} \\ &\quad \cdot \Psi_{r_j}^\top [(SP)^\top W_e SP + W_{\Delta f}] \Psi_{r_j} \\ L &= \left[ \Psi_{r_j}^\top ((SP)^\top W_e SP + W_f + W_{\Delta f}) \Psi_{r_j} \right]^{-1} \\ &\quad \cdot \left[ \Psi_{r_j}^\top (SP)^\top W_e \right] \end{aligned}$$

with  $\Psi_{r_j} = \Psi r_j$ . This solution follows from evaluating the necessary condition for optimality  $\frac{\partial V}{\partial \theta_{j+1}} = 0$ . Next, a construction approach for non-causal ROBF is presented.

#### 3.3 A Construction Approach for Non-Causal ROBF

In this subsection, an approach is presented for the generation of ROBF in  $\mathcal{RL}_2$ . Here, pole selection plays an important role. Often in control and identification, the prespecified poles are located in  $\mathbb{D}$ , see, e.g., Heuberger et al. (2005), Ninness and Gustafsson (1997), and Boche and Pohl (2007). Crucially, here it is proposed to extend this framework with prespecified poles in  $\mathbb{E}$ . For ILC, this enables the use of infinite pre-actuation, i.e., non-causal control actions, which is key to the effectiveness of ILC.

The orthonormal basis functions considered here are generated by two periodic sequences, defined by the sets of poles  $\{\xi_{s,i}\}_{i=1,\dots,n_s}$  and  $\{\xi_{u,i}\}_{i=1,\dots,n_u}$ , with  $\xi_{s,i}, \xi_{u,i} \in \mathbb{D}$ ,  $\forall i$ . These poles can be cyclically repeated, i.e.,  $\xi_{s,i+n_s} = \xi_{s,i}$  and  $\xi_{u,i+n_u} = \xi_{u,i}$ . The basis functions are given by

$$\psi_{s,i}(z) = \frac{\sqrt{1 - |\xi_{s,i}|^2}}{z - \xi_{s,i}} \phi_i(z, \xi_s), \quad (9)$$

$$\psi_{u,i}(z) = \frac{\sqrt{1 - |\xi_{u,i}|^2}}{1 - \xi_{u,i} z} \frac{1}{\phi_i(z, \xi_u)}, \quad (10)$$

where the so-called Blaschke products  $\phi_i$  are defined by

$$\phi_i(z, \xi) = \begin{cases} 1 & \text{if } i = 1, \\ \prod_{m=1}^{i-1} \frac{1 - \overline{\xi_m} z}{z - \xi_m} & \text{if } i > 1. \end{cases}$$

The set  $\{\psi_{s,i}\}_{i>0} \in \mathcal{H}_{2-}$  are the Takenaka-Malmquist functions, and contains causal basis functions. The set  $\{\psi_{u,i}\}_{i>0} \in \mathcal{H}_{2-}^\perp$  consists of all non-causal functions. Together, these form the basis functions to be used:

$$\Psi(z) = [\psi_{s,1}(z), \psi_{s,2}(z), \dots, \psi_{u,1}(z), \psi_{u,2}(z), \dots]^\top.$$

Here, the space  $\mathcal{H}_{2-}$  denotes all functions in  $\mathcal{H}_2$  that are zero at infinity, such as strictly proper stable systems.  $\mathcal{H}_{2-}^\perp$  denotes the orthogonal complement of  $\mathcal{H}_{2-}$  in  $\mathcal{L}_2$ .

In view of the user-defined sets of poles  $\{\xi_{s,i}\}_{i=1,\dots,n_s}$  and  $\{\xi_{u,i}\}_{i=1,\dots,n_u}$ , the following observations are made, which can be interpreted as guidelines for pole selection:

- The linear span of the basis functions  $\Psi(e^{j\omega})$  is dense in  $\mathcal{L}_p$ ,  $1 < p < \infty$ , or  $\mathcal{L}_p$  complete, if and only if

$$\sum_{i=1}^{\infty} (1 - |\xi_{s,i}|) = \infty, \text{ and } \sum_{i=1}^{\infty} (1 - |\xi_{u,i}|) = \infty. \quad (11)$$

In other words, any orthonormal basis satisfying (11) can arbitrarily well model any system in  $\mathcal{L}_p$  (Achiesser, 1992; Ninness and Gustafsson, 1997; Akçay, 2000).

- The undermodeling error  $|P^{-1}(e^{i\omega}) - F(e^{i\omega}, \theta^*)|$ , with  $F(z, \theta^*)$  the best  $\mathcal{L}_2$  approximation of  $P^{-1}(z)$ , can be decreased by choosing the poles  $\{\xi_{s,i}\}_{i=1,\dots,n_s}$  and  $\{1/\xi_{u,i}\}_{i=1,\dots,n_u}$  closer to the true poles of  $P^{-1}(z)$  (Ninness et al., 1999). In fact, if the poles of  $P^{-1}$  are perfectly known, then i) the error reduces to zero, and ii) no cyclic repetition of the poles is needed.
- In case  $P^{-1}(z)$  contains direct feedthrough, the choice  $\xi_{u,1} = 0$  in (10) can be used to generate a basis function  $\psi_{u,1} = 1$ . If  $P^{-1}(z)$  is non-proper, additional steps of preview can be added, such that a perfect parametrization is constructed.
- When complex poles are incorporated, the basis functions (9) and (10) must be adapted to ensure real-valued impulse responses. Real-valued basis functions are obtained through unitary transformations of (9) and (10), see Ninness and Gustafsson (1997), Akçay (2000), with the obvious requirement that complex poles always occur in complex conjugated pairs.

**Remark 1.** Depending on the order in which the poles are included in (9) and (10), different bases are obtained. Bodin et al. (2000) present methods for optimal ordering of the poles for several criteria.

Summarizing, in practice the pole sets  $\{\xi_{s,i}\}_{i=1,\dots,n_s}$  and  $\{1/\xi_{u,i}\}_{i=1,\dots,n_u}$  should be chosen as close as possible to the true poles of  $P^{-1}$ . To further reduce possible undermodeling, these poles can be cyclically repeated.

### 3.4 Connections with Pre-Existing Approaches to ILC with Basis Functions

In the previous subsections, a new approach is proposed to ILC with basis functions that combines the advantages of pre-existing approaches: a convex optimization problem, and a rational feedforward controller. In view of these as-

pects, in this subsection connections are provided between the proposed approach and earlier approaches.

Crucially, the pre-existing approaches in Van de Wijdeven and Bosgra (2010), Bolder et al. (2014, 2015), Blanken et al. (2016), and Van Zundert et al. (2016a) use basis functions  $\psi(z)$  that are Laurent polynomials in  $z$ , i.e.  $\psi(z) \in \mathbb{R}[z, z^{-1}]$ , rather than general rational functions.

*Linear Feedforward Parametrization* The approach in Van de Wijdeven and Bosgra (2010), Bolder et al. (2014) employs a linear combination, see Definition 2, of such Laurent polynomials. Therefore, this approach is recovered as a special case of the proposed methodology in this paper by fixing the poles  $\{\xi_s\}$  and  $\{\xi_u\}$  of the rational basis functions (9) and (10) at  $z = 0$ . Hence, although the approach in Van de Wijdeven and Bosgra (2010) and Bolder et al. (2014) admits the same analytic solution expression (8) as the proposed methodology, the achievable performance is limited due to the restricted pole locations. Furthermore, the amount of pre-actuation and post-actuation is finite and limited by the order of the Laurent polynomials.

*Rational Feedforward Parametrization* Rather than using a linear combination of basis functions, the approaches in Bolder et al. (2015), Blanken et al. (2016), and Van Zundert et al. (2016a) employ a rational feedforward parametrization, as defined next.

**Definition 3** (Rational feedforward parametrization). *The feedforward controller  $F(\theta)$  parameterized in terms of a rational basis  $\mathcal{F}_{rat}$  is given by*

$$\mathcal{F}_{rat} = \left\{ \frac{A(\theta)}{B(\theta)} \mid \theta \in \mathbb{R}^{n_\theta} \right\}$$

with

$$A(z, \theta) = \sum_{i=1}^{n_a} \theta[i] \psi_i(z), \quad B(z, \theta) = 1 + \sum_{i=n_a+1}^{n_a+n_b} \theta[i] \psi_i(z),$$

with basis functions  $\psi_i(z)$ ,  $i = 1, \dots, n_\theta$ , and  $n_\theta = n_a + n_b$ .

Importantly, this parametrization imposes no restrictions on the poles of  $F(\theta)$ . In fact, the poles are part of the optimization problem, and can be placed in  $\mathbb{E}$ . Using the same stable inversion method as described in Subsection 3.1, this allows for infinite pre-actuation. Similarly, poles in  $\mathbb{D}$  enable infinite post-actuation. However, as a consequence of free poles, the optimization problem is nonlinear. Often, iterative algorithms are invoked, for which (global) optimality cannot be guaranteed in general, see, e.g., Bolder et al. (2015) and Van Zundert et al. (2016a).

**Remark 2.** *Parametrization  $\mathcal{F}_{rat}$  is a direct extension of joint input shaping and feedforward (Boeren et al., 2014), which for the specific class of point-to-point motion tasks retains a convex optimization problem.*

Summarizing, the approach in Van de Wijdeven and Bosgra (2010) and Bolder et al. (2014) is recovered as a special case of the proposed methodology in this paper, and is restricted in the achievable performance. The approaches using a rational parametrization in Bolder et al. (2015), Blanken et al. (2016), and Van Zundert et al. (2016a) allow for high performance through pole optimization, but require nonlinear optimization algorithms.

Table 1. Parameters of the flexible cart system.

parameter	symbol	true value	value for ILC	unit
mass	$m$	1	0.9	kg
inertia	$I$	$1.667 \times 10^{-3}$	$1.5 \times 10^{-3}$	$\text{kgm}^2$
spring constant	$k$	$1 \times 10^4$	$1.1 \times 10^4$	N/m
damping constant	$d$	10	5	Ns/m
length	$l$	0.1	0.1	m

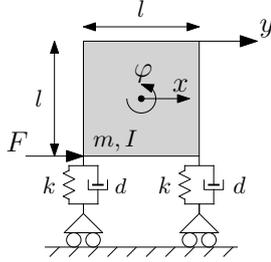


Fig. 3. The flexible cart system, with input force  $F$  and output position  $y$ , has translation and rotation freedom in  $x$  and  $\varphi$ , respectively.

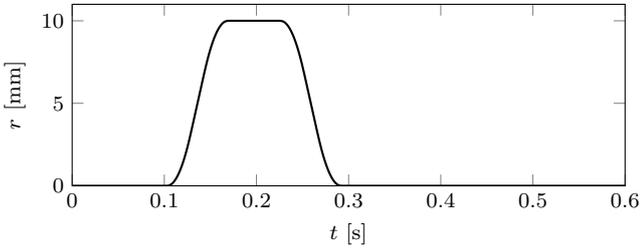


Fig. 4. The reference signal  $r$  is a fourth-order motion task.

#### 4. SIMULATION EXAMPLE

In this section, the proposed methodology using rational basis functions is validated, and compared to pre-existing approaches using Laurent polynomials. In particular, the benefits of non-causal feedforward and infinite pre- and post-actuation are demonstrated.

Consider the mechanical system shown in Figure 3, see also Van Zundert et al. (2016b), with parameters in Table 1. The system is linearized around  $\varphi = 0$ , and discretized using zero-order-hold on the input with sampling time 1 ms. The resulting plant and feedback controller are

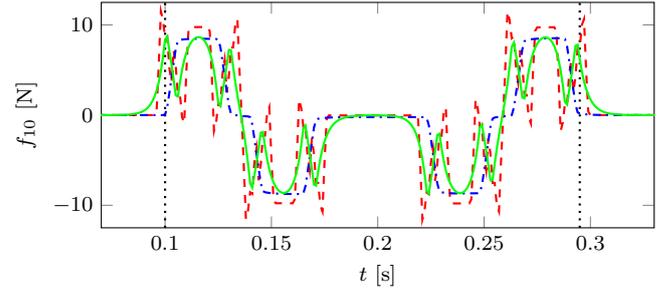
$$P(z) = \frac{-2.407 \times 10^{-7}(z + 0.9707)(z - 0.8051)(z - 1.319)}{(z - 1)^2(z^2 - 1.941z + 0.9704)},$$

$$C(z) = \frac{426.0(z - 0.9854)(z + 1)}{(z - 0.8762)(z - 0.8273)}.$$

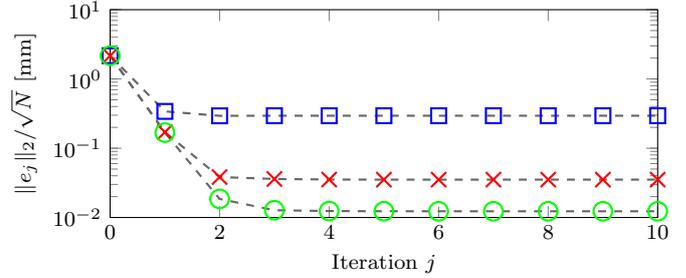
Here, it is crucial to note that  $P(z)$  has one zero in  $\mathbb{E}$ . The system repeatedly performs the reference signal  $r$  of  $N = 601$  samples, shown in Figure 4. The following approaches to ILC with basis functions are compared:

- The proposed approach to ILC with both non-causal and causal rational basis functions, see Section 3.
- ILC using only the subset of causal rational basis functions, i.e., in  $\mathcal{H}_2$ .
- ILC with Laurent polynomials, see Subsection 3.4.

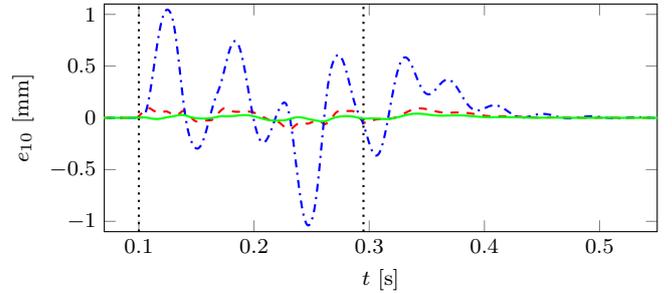
All ILC approaches use the same model for  $SP$ , see (3), which is constructed using the parameters for ILC in Table 1. The rational basis functions (9) and (10) are generated based on the zeros of the model of  $P$ , with



(a) Feedforward signals  $f_{10}$ , illustrating pre- and post-actuation.



(b) Sample-normalized two-norm of error signals.



(c) Error signals  $e_{10}$ .

Fig. 5. The proposed approach with non-causal rational basis functions (—,  $\odot$ ) outperforms the pre-existing approaches which employ the subset of causal rational basis functions (---,  $\square$ ) and Laurent polynomials (---,  $\times$ ). In particular, infinite pre-actuation and post-actuation is generated, while causal functions only enable post-actuation, and Laurent polynomials generate only finite pre-actuation and post-actuation.

$\{\xi_{s,i}\} = \{-0.9837, 0.7752\}$  and  $\{\xi_{u,i}\} = \{1/1.3337\}$ . Here, it is important to notice that these zeros do not coincide with the true zeros of the plant  $P(z)$ . The used Laurent polynomials are  $\psi_i = z^{-k}$ , with  $k = [-5, -4, \dots, 9, 10]$ . To show the benefit of appropriately selecting the poles, the amount of Laurent polynomials is chosen significantly higher than the number of used rational basis functions. The weighting matrices in (4) are selected as  $W_e = I$ ,  $W_f = 10^{-9}I$  and  $W_{\Delta f} = 0$ . The results are presented in Figure 5. The following observations are made:

- The proposed approach using non-causal rational basis functions outperforms the approach using only the subset of causal basis functions. This demonstrates the importance of explicitly compensating the non-minimum phase zero of  $P$  through pre-actuation.
- The rational basis functions enable infinite pre- and post-actuation. In sharp contrast, the amount of pre- and post-actuation with Laurent polynomials is inherently finite, which restricts performance.

## 5. CONCLUSIONS

In this paper, a new approach for ILC with basis functions is presented, which significantly enhances existing approaches. At present, high performance for non-repeating tasks comes at the cost of non-convex optimization. The main contribution of this paper is an approach to ILC using rational orthonormal basis functions in  $\mathcal{L}_2$ , which i) retains a convex optimization problem, ii) enables compensation for general rational systems, and iii) utilizes infinite pre-actuation and post-actuation. This approach is in sharp contrast with the use of rational orthonormal basis functions in system identification, which is typically focused on parametrizing models in  $\mathcal{H}_2$ . A simulation study demonstrates the benefits of non-causal basis functions in ILC, including infinite preview.

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