Model-Free Learning for Massive MIMO Systems:
Stochastic Approximation Adjoint Iterative Learning Control*

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Abstract—**Learning can substantially increase the performance of control systems that perform repeating tasks. The aim of this paper is to develop an efficient iterative learning control algorithm for MIMO systems with a large number of inputs and outputs that does not require model knowledge. The gradient of the control criterion is obtained through dedicated experiments on the system. Using a judiciously selected randomization technique, an unbiased estimate of the gradient is obtained from a single dedicated experiment, resulting in fast convergence of a Robbins-Monro type stochastic gradient descent algorithm. Analysis shows that the approach is superior to earlier deterministic approaches and to related SPSA-type algorithms. The approach is illustrated on a multivariable example.**

I. INTRODUCTION

The aim of iterative learning control (ILC) is to achieve high performance while providing robustness against model errors. Typical frameworks include Arimoto-type ILC [1], frequency domain ILC [2], [3], and optimization-based approaches such as norm-optimal ILC [4], [5]. Commonly, an approximate model is used in conjunction with measured data, and a contraction map is used to guarantee convergence.

In [6], an optimization-based ILC algorithm is introduced that employs dedicated experiments on the system instead of a model. The essential idea is to do experiments on the adjoint system [7] to obtain the gradient of a cost criterion, leading to a gradient descent ILC algorithm [8], see also [9] for a case study. In sharp contrast to model-based norm-optimal ILC, or data-driven norm-optimal ILC algorithms that estimate a model using data from previous iterations [10], this enables a complete data-driven design for MIMO systems without the need for modeling. Similar approaches have been developed for MIMO experiment-based virtual reference feedback tuning [11], iterative feedback tuning (IFT) [12], and $\mathcal{H}_\infty$-norm estimation [13]. These approaches generate the gradient of the system through dedicated experiments, typically requiring $n_i \times n_o$ experiments per iteration for an n_i by n_o system.

Although model-free adjoint ILC removes the need for models in ILC, it does not scale well for massive MIMO systems because of the high number of dedicated experiments. The aim of this paper is to develop an efficient algorithm using an unbiased gradient estimate obtained from a single experiment, which is used in conjunction with stochastic approximation. In addition, the use of a serial ILC implementation instead of the common parallel implementation is advocated to enhance the convergence speed of gradient descent ILC algorithms. The contribution is fourfold.

1) A new approach to adjoint ILC for massive MIMO systems is introduced that requires a single experiment at each iteration through a Robbins-Monro type stochastic gradient descent algorithm (Section III).
2) The presented approach is theoretically compared to deterministic approaches for symmetric and non-symmetric MIMO systems, and to a direct implementation of related SPSA-type algorithms (Section IV).
3) A serial ILC implementation is presented to enhance the convergence speed of gradient descent ILC algorithms for closed-loop systems (Section V).
4) The proposed framework is illustrated using a multi-axis industrial flatbed printer model (Section VI).

The developed approach is a stochastic optimization algorithm and relates to simultaneous perturbation stochastic approximation (SPSA). In stochastic optimization, a cost criterion is minimized using noisy measurements or approximations of the gradient [14], [15]. The Robbins-Monro algorithm [16] uses noisy measurements of the gradient of the cost criterion for gradient descent minimization, and has been employed in ILC, e.g. in [17] and in [18], where the influence of disturbances on SISO model-free adjoint ILC is analyzed. The Kiefer-Wolfowitz algorithm [19] considers a situation where the gradient cannot be measured directly, and instead has to be estimated from noisy cost criterion evaluations. In [20], SPSA is introduced, where instead of the gradient an estimated derivative in a random combination of input directions is used for the parameter update, which reduces the required number of cost criterion evaluations significantly. Such an SPSA approach is used in [21] in a randomized IFT algorithm. In the present paper, it is shown that SPSA does not lead to the desired efficiency in adjoint ILC, and a fundamentally different approach is developed.

II. GRADIENT DESCENT ILC

In this section, the considered gradient descent ILC problem is formulated. Consider a MIMO system J with n_i

inputs and n_o outputs, given in lifted form by

$$\underbrace{\begin{bmatrix} y^1 \\ \vdots \\ y^{n_o} \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} J^{11} & \cdots & J^{1n_i} \\ \vdots & & \vdots \\ J^{n_o 1} & \cdots & J^{n_o n_i} \end{bmatrix}}_{J} \underbrace{\begin{bmatrix} f^1 \\ \vdots \\ f^{n_i} \end{bmatrix}}_{f} \tag{1}$$

$$\underbrace{\begin{bmatrix} e^1 \\ \vdots \\ e^{n_o} \end{bmatrix}}_{e} = \underbrace{\begin{bmatrix} r^1 \\ \vdots \\ r^{n_o} \end{bmatrix}}_{r} - \underbrace{\begin{bmatrix} y^1 \\ \vdots \\ y^{n_o} \end{bmatrix}}_{y} \tag{2}$$

with input f, output y, error e and unknown exogenous disturbance r. Here, $J^{lm} \in \mathbb{R}^{N \times N}$ for finite signal length $N \in \mathbb{Z}^+$, and $y^l, e^l, r^l, f^m \in \mathbb{R}^{N \times 1}$ for $l = 1, ..., n_o$, $m = 1, ..., n_i$. Both open-loop and closed-loop systems can be represented by J, see Section V.

The criterion

$$\mathcal{J}(f) = \|e\|_{W_e}^2 + \|f\|_{W_f}^2 \tag{3}$$

is minimized, where $\|x\|_W = \sqrt{x^\top W x}$, and $W_e \succ 0$ and $W_f \succeq 0$ are weight matrices. The optimal input is given by $f^* = \arg\min_f \mathcal{J}(f)$. Criterion (3) is minimized using an iterative gradient descent algorithm. The parameter update for iteration j is given by

$$f_{j+1} = f_j - \varepsilon_j g(f_j), \tag{4}$$

with step size ε_j and gradient $g(f_j) = \frac{\partial \mathcal{J}}{\partial f}(f_j)$ as given in the following lemma.

Lemma 1. *The gradient $g(f_j)$ of $\mathcal{J}(f)$ is given by*

$$g(f_j) = -2J^\top W_e e_j + 2W_f f_j. \tag{5}$$

Experiments on $J^\top$ can be done directly for SISO systems, by noting that $J^\top$ is the adjoint operator of J and relates to J through a time reversal, as described in the following.

Definition 1 (Adjoint). *The inner product of two signals $f, g \in \mathbb{R}^{N \times 1}$ is given by $\langle f, g \rangle = f^\top g$. The adjoint J^* of J is defined as the operator that satisfies the condition*

$$\langle f, Jg \rangle = \langle J^* f, g \rangle, \quad \forall f, g \in \mathbb{R}^{N \times 1}.$$

The adjoint J^* of J is given by $J^\top$, which follows from

$$f^\top J g = (J^* f)^\top g = f^\top (J^*)^\top g, \quad \forall f, g \in \mathbb{R}^{N \times 1}.$$

Lemma 2. *The adjoint of a SISO system $J = J^{11}$ is given by $\left(J^{11}\right)^\top = \mathcal{T} J^{11} \mathcal{T}$, where the involutory permutation matrix*

$$\mathcal{T} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & 1 & 0 \\ 0 & \iddots & & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{N \times N}$$

has the interpretation of a time-reversal operator.

For SISO systems, Lemma 2 enables the measurement of the gradient using a single experiment by performing two time reversals. However, this is not applicable to non-symmetric MIMO systems, as is shown next.

Lemma 3. *The adjoint of a MIMO system J is given by*

$$J^\top = \begin{bmatrix} (J^{11})^\top & \cdots & (J^{n_o 1})^\top \\ \vdots & & \vdots \\ (J^{1n_i})^\top & \cdots & (J^{n_o n_i})^\top \end{bmatrix} \tag{6}$$

$$= \underbrace{\begin{bmatrix} \mathcal{T} & & 0 \\ & \ddots & \\ 0 & & \mathcal{T} \end{bmatrix}}_{\mathcal{T}^{n_i}} \underbrace{\begin{bmatrix} J^{11} & \cdots & J^{n_o 1} \\ \vdots & & \vdots \\ J^{1n_i} & \cdots & J^{n_o n_i} \end{bmatrix}}_{\tilde{J}} \underbrace{\begin{bmatrix} \mathcal{T} & & 0 \\ & \ddots & \\ 0 & & \mathcal{T} \end{bmatrix}}_{\mathcal{T}^{n_o}}.$$

Since $\tilde{J} \neq J$ for non-symmetric MIMO systems, the term $J^\top e_j$, and therefore the gradient (5), cannot be determined from a single experiment on J. Instead, the gradient $g(f_j)$ can be determined exactly from $n_i \times n_o$ judiciously chosen experiments, as is explained in Section IV.

The aim of this paper is to develop an efficient algorithm using an unbiased gradient estimate obtained from a single experiment. This algorithm is introduced in the next section.

III. STOCHASTIC APPROXIMATION ADJOINT ILC

In this section the stochastic approximation adjoint ILC (SAAILC) algorithm is introduced. First, the main idea is explained using an example of a 2x2 system. Secondly, the approach is formalized. Thirdly, the implementation is explained and lastly, a convergence analysis is given.

The main idea of this paper is to use an approximation of the gradient instead of the deterministic gradient (5) in the parameter update (4), since obtaining (5) deterministically is experimentally expensive for non-symmetric MIMO systems. The approximation is chosen such that it is experimentally exceptionally advantageous, i.e., only a single experiment is required per gradient descent iteration. Additionally, the approximation is chosen such that it is unbiased and a convergence proof can be derived.

The essential new step is the introduction of the matrix $A_j \in \mathbb{R}^{(N n_i) \times (N n_o)}$, given by

$$A_j = \begin{bmatrix} a_j^{11} & \cdots & a_j^{1n_o} \\ \vdots & \ddots & \vdots \\ a_j^{n_i 1} & \cdots & a_j^{n_i n_o} \end{bmatrix} \otimes I_N \tag{7}$$

where I_N is the $N \times N$ identity matrix and the entries a_j^{lm} are samples from a symmetric Bernoulli ± 1 distribution, i.e., $a_j^{lm} \in \{-1, 1\}$ and the probabilities are given by $P(a_j^{lm} = 1) = 1/2$ and $P(a_j^{lm} = -1) = 1/2$. This matrix can be used to obtain an unbiased gradient approximation from a single experiment, as shown in the following example.

A. Main idea: 2x2 example

Consider a system J with two inputs and two outputs and take $W_e = W_f = I$, such that

$$g(f_j) = -2 \begin{bmatrix} (J^{11})^\top e_j^1 + (J^{21})^\top e_j^2 \\ (J^{12})^\top e_j^1 + (J^{22})^\top e_j^2 \end{bmatrix} + 2 \begin{bmatrix} f_j^1 \\ f_j^2 \end{bmatrix}. \tag{8}$$

The gradient (8) can be determined as in [6] through four dedicated experiments, one for each of the terms $(J^{lm})^\top e_j^l$. The main idea of this paper is that an unbiased estimate

$\hat{g}(f_j)$ of the gradient $g(f_j)$ can be obtained for a given f_j from a single experiment as

$$\hat{g}(f_j) = -2\mathcal{T}^{n_i} A_j J A_j \mathcal{T}^{n_o} e_j + 2 f_j. \qquad (9)$$

The term $\mathcal{T}^{n_i} A_j J A_j \mathcal{T}^{n_o} e_j$ in (9) follows from a single experiment on J, where A_j is generated according to (7). In the experiment the linear transformation $\mathcal{T}^{n_o}$ is applied to the system input, $\mathcal{T}^{n_i}$ is applied to the output and A_j is applied to both input and output. Application of A_j to the system input results in

$$\mathcal{T}^{n_i} J A_j \mathcal{T}^{n_o} e_j = \qquad (10)$$
$$\begin{bmatrix} (J^{11})^\top (a_j^{11} e_j^1 + a_j^{12} e_j^2) + (J^{12})^\top (a_j^{21} e_j^1 + a_j^{22} e_j^2) \\ (J^{21})^\top (a_j^{11} e_j^1 + a_j^{12} e_j^2) + (J^{22})^\top (a_j^{21} e_j^1 + a_j^{22} e_j^2) \end{bmatrix}.$$

Application of A_j to both system input and output gives

$$\mathcal{T}^{n_i} A_j J A_j T^{n_o} e_j = \qquad (11)$$
$$\begin{bmatrix} a_j^{11}((J^{11})^\top (a_j^{11} e_j^1 + a_j^{12} e_j^2) + (J^{12})^\top (a_j^{21} e_j^1 + a_j^{22} e_j^2)) \\ a_j^{21}((J^{11})^\top (a_j^{11} e_j^1 + a_j^{12} e_j^2) + (J^{12})^\top (a_j^{21} e_j^1 + a_j^{22} e_j^2)) \\ + a_j^{12}((J^{21})^\top (a_j^{11} e_j^1 + a_j^{12} e_j^2) + (J^{22})^\top (a_j^{21} e_j^1 + a_j^{22} e^2)) \\ + a_j^{22}((J^{21})^\top (a_j^{11} e_j^1 + a_j^{12} e_j^2) + (J^{22})^\top (a_j^{21} e_j^1 + a_j^{22} e_j^2)) \end{bmatrix}.$$

All terms in (11) contain a multiplier $a_j^{\alpha\beta} a_j^{\gamma\delta}$. It holds that $a_j^{\alpha\beta} a_j^{\gamma\delta} = 1$ if $\alpha = \gamma, \beta = \delta$. Otherwise, it holds that $P(a_j^{\alpha\beta} a_j^{\gamma\delta} = 1) = 1/2$ and $P(a_j^{\alpha\beta} a_j^{\gamma\delta} = -1) = 1/2$, such that $\mathbb{E}(a_j^{\alpha\beta} a_j^{\gamma\delta}) = 0$ when $\alpha \neq \gamma$ or $\beta \neq \delta$. Therefore, (11) is equal to $\tilde{y}_j + \eta_j$, with

$$\tilde{y}_j = \begin{bmatrix} (J^{11})^\top e_j^1 + (J^{21})^\top e_j^2 \\ (J^{12})^\top e_j^1 + (J^{22})^\top e_j^2 \end{bmatrix}, \qquad (12)$$

and η_j all remaining terms from (11) for which $\alpha \neq \gamma$ or $\beta \neq \delta$. Since $\mathbb{E}(a_j^{\alpha\beta} a_j^{\gamma\delta}) = 0$ when $\alpha \neq \gamma$ or $\beta \neq \delta$, it holds that $\mathbb{E}(\eta_j) = 0$ and therefore $\mathbb{E}(\mathcal{T}^{n_i} A_j J A_j T^{n_o} e_j) = \tilde{y}_j$. Since $\tilde{y}_j$ is equal to the term $J^\top e_j$ in the gradient (5),

$$\mathbb{E}(\hat{g}(f_j)) = g(f_j). \qquad (13)$$

Thus, (9) gives an unbiased estimate of the gradient, which is an essential step in the gradient descent parameter update.

B. SAAILC for massive MIMO systems

The approach shown in the previous subsection is formalized for general MIMO systems in the following theorem.

Theorem 1. *Let*

$$\hat{g}(f_j) = -2\mathcal{T}^{n_i} A_j J A_j \mathcal{T}^{n_o} W_e e_j + 2 W_f f_j. \qquad (14)$$

Then,

$$\mathbb{E}(\hat{g}(f_j)) = g(f_j). \qquad (15)$$

Proof. Since the term $W_f f_j$ is known, only the experimental term $\mathcal{T}^{n_i} A_j J A_j \mathcal{T}^{n_o} W_e e_j$ is considered. Denote $\tilde{e}_j = W_e e_j$. It holds that

$$\mathcal{T}^{n_i} A_j J A_j \mathcal{T}^{n_o} \tilde{e}_j = \begin{bmatrix} \sum_{k=1}^{n_o} \left(\sum_{l=1}^{n_o} a_j^{1l} \sum_{m=1}^{n_i} (J^{lm})^\top a_j^{mk} \right) \tilde{e}_j^k \\ \vdots \\ \sum_{k=1}^{n_o} \left(\sum_{l=1}^{n_o} a_j^{n_i l} \sum_{m=1}^{n_i} (J^{lm})^\top a_j^{mk} \right) \tilde{e}_j^k \end{bmatrix}. \qquad (16)$$

Since $a_j^{\alpha\beta} a_j^{\gamma\delta} = 1$ if $\alpha = \gamma, \beta = \delta$ and $\mathbb{E}(a_j^{\alpha\beta} a_j^{\gamma\delta}) = 0$ otherwise, (16) is equal to $\tilde{y}_j + \eta_j$, with

$$\tilde{y}_j = \begin{bmatrix} \sum_{k=1}^{n_o} (J^{k1})^\top \tilde{e}_j^k \\ \vdots \\ \sum_{k=1}^{n_o} (J^{kn_i})^\top \tilde{e}_j^k \end{bmatrix}, \qquad (17)$$

and η_j all remaining terms from (16) for which $\alpha \neq \gamma$ or $\beta \neq \delta$. Note that $\tilde{y}_j = J^\top W_e e_j$ in (5). Since $\mathbb{E}(a_j^{\alpha\beta} a_j^{\gamma\delta}) = 0$ if $\alpha \neq \gamma$ or $\beta \neq \delta$, it holds that $\mathbb{E}(\eta_j) = 0$ and therefore

$$\mathbb{E}(\mathcal{T}^{n_i} A_j J A_j \mathcal{T}^{n_o} W_e e_j) = \tilde{y}_j = J^\top W_e e_j. \qquad (18)$$

Thus, (15) holds. □

The main idea is that (14) requires only one experiment to obtain an unbiased approximation of the gradient, independent of the number of inputs and outputs. This approximation is used to update the input signal as

$$f_{j+1} = f_j - \varepsilon_j \hat{g}(f_j). \qquad (19)$$

Using Theorem 1, the estimate $\hat{g}(f_j)$ is rewritten as

$$\hat{g}(f_j) = g(f_j) + \eta_j, \qquad (20)$$

where $\mathbb{E}(\eta_j) = 0$ by Theorem 1.

C. Implementation of the SAAILC algorithm

The implementation of SAAILC is outlined in Algorithm 1, which shows that each iteration requires only a single additional experiment to obtain an unbiased estimate of the gradient, regardless of the number of inputs and outputs.

Algorithm 1 Stochastic Approximation Adjoint ILC

1: **for** $j = 1 : n_{iteration}$
2:    Apply input f_j and measure $e_j = r - J f_j$.
3:    Time reverse: $\bar{e}_j = \mathcal{T}^{n_o} W_e e_j$.
4:    Generate the matrix A_j in (7).
5:    Compute $A_j \bar{e}_j$ and apply to measure $J A_j \bar{e}_j$.
6:    Premultiply with A_j to obtain $\bar{z}_j = A_j J A_j \bar{e}_j$.
7:    Time reverse to obtain $\mathcal{T}^{n_i} \bar{z}_j$ and compute $\hat{g}_j$ in (14).
8:    Choose ε_j to meet Assumption 2.
9:    Update: $f_{j+1} = f_j - \varepsilon_j \hat{g}(f)$.
10: **end**

D. Convergence analysis

The SAAILC algorithm can be interpreted as a Robbins-Monro type algorithm of the form

$$f_{j+1} = f_j - \varepsilon_j (g(f_j) + \eta_j), \qquad (21)$$

where η_j is interpreted as a disturbance term. The following assumptions are made.

Assumption 1. *The iterates f_j remain almost surely bounded.*

Assumption 2. *The step size ε_j is chosen such that*

$$\sum_j^\infty \varepsilon_j = \infty, \qquad \sum_j^\infty \varepsilon_j^2 < \infty.$$

Assumption 1 and 2 can be satisfied by choosing ε_j appropriately, and lead to the following convergence theorem.

Theorem 2. *Under Assumption 1 and 2, the sequence of iterates $\{f_j\}$ in (21) converges to the minimizer f^* of $\mathcal{J}(f)$ almost surely.*

Note that $\mathcal{J}(f)$ is quadratic, and since $W_e \succ 0$ and $W_f \succeq 0$, it is nonnegative. Therefore, $\mathcal{J}(f)$ has one stationary point f^* that is the global minimum, and the gradient $g(f)$ of $\mathcal{J}(f)$ is Lipschitz. In addition, it holds that the disturbance term η_j is a Martingale difference sequence since $\mathbb{E}(\eta_j) = 0$, and it is square integrable. The proof of Theorem 2 follows from these properties and the almost sure convergence of a Robbins-Monro algorithm under Assumption 1 and 2, which is well covered in e.g. [22, Ch. 2].

IV. ALTERNATIVE APPROACHES

In this section, the proposed SAAILC algorithm is compared to symmetric and non-symmetric deterministic model-free MIMO adjoint ILC, and to an SPSA-like approach.

A. Symmetric deterministic MIMO adjoint ILC

In the symmetric deterministic MIMO adjoint ILC approach, the term $J^\top e_j$ in the gradient (5) is measured as

$$J^\top e_j = \mathcal{T}^{n_i} J \mathcal{T}^{n_o} e_j, \qquad (22)$$

i.e., it is assumed that $\tilde{J} = J$ (cf. (6)). This assumption can only be made for square systems, and holds for symmetric MIMO systems. For non-symmetric systems, it can lead to reduced performance or divergence of the ILC algorithms.

B. Non-symmetric deterministic MIMO adjoint ILC

A general deterministic MIMO adjoint ILC algorithm for non-symmetric systems is introduced in [6]. The term $J^\top e_j$ in the gradient (5) can be determined exactly using $n_i \times n_o$ dedicated experiments that are structured as

$$J^\top e_j = \mathcal{T}^{n_i} \left(\sum_{l=1}^{n_i} \sum_{m=1}^{n_o} E^{lm} J E^{lm} \right) \mathcal{T}^{n_o} e_j, \qquad (23)$$

where E^{lm} consists of zeros, with a one on the lm^{th} entry. Since (23) requires $n_i \times n_o$ dedicated experiments for each iteration to obtain $g(f_j)$ deterministically, this approach is experimentally infeasible for massive MIMO systems.

C. SPSA-like algorithms

In simultaneous perturbation stochastic approximation (SPSA) for quadratic cost functions, an unbiased gradient estimate is obtained from the derivative in a random combined input direction [15]. In [21], an SPSA-like approach for iterative feedback tuning for MIMO systems is suggested to reduce the required number of experiments per iteration.

To apply SPSA to adjoint ILC, the deterministic derivative in a random combined input direction is required. However,

this cannot be obtained directly from a single experiment. The derivative of $\mathcal{J}(f)$ to a single input f_j^m is given by

$$\frac{\partial \mathcal{J}(f)}{\partial f_j^m} = -2 \sum_{l=1}^{n_o} (J^{lm})^\top \tilde{e}_j^l + 2W_f f_j^m. \qquad (24)$$

which requires n_o experiments. Determining a deterministic derivative in a random combined direction of all inputs f_j^m, $m = 1, ..., n_i$ thus typically requires $n_i \times n_o$ experiments. Therefore, the use of an SPSA-like approach does not reduce the required number of experiments per iteration for the MIMO adjoint ILC problem considered in this paper.

V. CLOSED-LOOP ASPECTS: SERIAL ILC

In this section, a serial ILC structure is advocated. The main idea is that the convergence rate is directly affected by the choice of controller structure, which is investigated in detail.

The convergence rate of deterministic MIMO adjoint ILC depends on the step size ε_j, which is limited by the monotonic convergence condition $\varepsilon_j < \bar{\varepsilon}$ [6, Theorem 7], with

$$\bar{\varepsilon} = \| J^\top W_e J + W_f \|^{-1}. \qquad (25)$$

The maximum step size $\bar{\varepsilon}$ is a function of the system J and the weights W_e and W_f. The influence of J is analyzed in the frequency domain by taking $W_e = I$. Since the data-driven approach does not require robustness against model uncertainty, $W_f = 0$. Let $\sigma_n(X)$ and $\lambda_n(X)$ denote the nth singular value and eigenvalue of X, respectively. A z-domain interpretation of the deterministic algorithm (4) with $\varepsilon_j = \varepsilon$, $\forall j$ and $z = e^{i\omega}$ is given by [6, Sec. 3]

$$\mathbf{f}_{j+1}(z) = (I - 2\varepsilon \mathbf{J}^*(z)\mathbf{J}(z))\mathbf{f}_j(z) + 2\varepsilon \mathbf{J}^*(z)\mathbf{r}(z). \qquad (26)$$

The iteration (26) is monotonically convergent if

$$\bar{\sigma}(I - 2\varepsilon \mathbf{J}^*(e^{i\omega})\mathbf{J}(e^{i\omega})) < 1, \ \forall \omega \in [0, 2\pi). \qquad (27)$$

Matrix $\mathbf{J}^*(e^{i\omega})\mathbf{J}(e^{i\omega})$, $\mathbf{J}(e^{i\omega}) \in \mathbb{C}^{n_o \times n_i}$ is symmetric and nonnegative, such that $\mathbf{M}(e^{i\omega}) = (I - 2\varepsilon \mathbf{J}^*(e^{i\omega})\mathbf{J}(e^{i\omega}))$ is a symmetric matrix, and $\sigma_n(\mathbf{M}(e^{i\omega})) = |\lambda_n(\mathbf{M}(e^{i\omega}))|$. Therefore, condition (27) holds if $\varepsilon < \bar{\varepsilon}$ with

$$\bar{\varepsilon} = \left(\sup_{\omega \in [0, 2\pi)} \bar{\sigma} \left(\mathbf{J}^*(e^{i\omega})\mathbf{J}(e^{i\omega}) \right) \right)^{-1}. \qquad (28)$$

The term $\mathbf{M}(e^{i\omega})$ in (26) has the interpretation of a direction-dependent and frequency-dependent update rate of $\mathbf{f}(e^{i\omega})$. Iteration-to-iteration changes are large for directions and frequencies for which $\sigma_n(\mathbf{M}(e^{i\omega})) \approx 0$, and small for $\sigma_n(\mathbf{M}(e^{i\omega})) \approx 1$. By (27), $\bar{\sigma}(\mathbf{M}(e^{i\omega})) < 1$, hence the convergence rate at a certain frequency is limited by $\bar{\sigma}(\mathbf{M}(e^{i\omega}))$.

A typical and suitable choice for ε is $\varepsilon = 0.5\bar{\varepsilon}$, which leads to $\underline{\sigma}(\mathbf{M}(e^{i\omega})) = 0$ for $\{\omega | (\bar{\sigma}(\mathbf{J}^*(e^{i\omega})\mathbf{J}(e^{i\omega})))^{-1} = \bar{\varepsilon}\}$, i.e., the update rate is maximized at the peak singular value of $\mathbf{J}(e^{i\omega})$. The following result applies.

Lemma 4. *For $\varepsilon = 0.5\bar{\varepsilon}$,*

$$\bar{\sigma}(\mathbf{M}(e^{i\omega})) = 1 - \frac{\underline{\sigma}(\mathbf{J}^*(e^{i\omega})\mathbf{J}(e^{i\omega}))}{\sup_{\omega \in [0, 2\pi)} \bar{\sigma}(\mathbf{J}^*(e^{i\omega})\mathbf{J}(e^{i\omega}))}. \qquad (29)$$
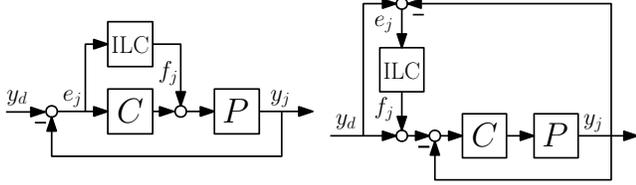
Fig. 1: Parallel (left) and serial (right) ILC implementations.

Proof. Note that from $\varepsilon = 0.5\bar{\varepsilon}$ and (28),

$$\sigma_n(-2\varepsilon\mathbf{J}^*(e^{i\omega})\mathbf{J}(e^{i\omega})) = \bar{\varepsilon}\sigma_n(\mathbf{J}^*(e^{i\omega})\mathbf{J}(e^{i\omega})) \quad (30)$$
$$= \frac{\sigma_n(\mathbf{J}^*(e^{i\omega})\mathbf{J}(e^{i\omega}))}{\sup_{\omega\in[0,2\pi)}\bar{\sigma}(\mathbf{J}^*(e^{i\omega})\mathbf{J}(e^{i\omega}))} \leq 1.$$

Since $(-2\varepsilon\mathbf{J}^*(e^{i\omega})\mathbf{J}(e^{i\omega}))$ is symmetric and non-positive, $\lambda_n(-2\varepsilon\mathbf{J}^*(e^{i\omega})\mathbf{J}(e^{i\omega})) = -\sigma_n(-2\varepsilon\mathbf{J}^*(e^{i\omega})\mathbf{J}(e^{i\omega}))$. Hence,

$$-1 \leq \lambda(-2\varepsilon\mathbf{J}^*(e^{i\omega})\mathbf{J}(e^{i\omega})) \leq 0. \quad (31)$$

Since $\mathbf{M}(e^{i\omega})$ is symmetric, it holds that $\sigma_n(\mathbf{M}(e^{i\omega})) = |\lambda_n(\mathbf{M}(e^{i\omega}))| = |1 + \lambda_n(-2\varepsilon\mathbf{J}^*(e^{i\omega})\mathbf{J}(e^{i\omega}))|$. Using (31), it follows that $\sigma_n(\mathbf{M}(e^{i\omega})) = 1 + \lambda_n(-2\varepsilon\mathbf{J}^*(e^{i\omega})\mathbf{J}(e^{i\omega})) = 1 - \sigma_n(-2\varepsilon\mathbf{J}^*(e^{i\omega})\mathbf{J}(e^{i\omega}))$. Hence, using (30),

$$\bar{\sigma}(\mathbf{M}(e^{i\omega})) = 1 - \frac{\sigma(\mathbf{J}^*(e^{i\omega})\mathbf{J}(e^{i\omega}))}{\sup_{\omega\in[0,2\pi)}\bar{\sigma}(\mathbf{J}^*(e^{i\omega})\mathbf{J}(e^{i\omega}))}. \quad (32)$$

$\square$

The key point is that system J is determined by the choice of ILC structure. In many ILC applications, including the deterministic MIMO adjoint ILC approach of [6], a parallel structure is used, see Fig. 1. The ILC input is applied as a feedforward signal between the controller and the plant, such that $J_{par} = P(I + CP)^{-1}$, with plant P and controller C. For frequencies where $\underline{\sigma}(\mathbf{P}(e^{i\omega})\mathbf{C}(e^{i\omega})) \gg 1$, $\bar{\sigma}(\mathbf{P}(e^{i\omega})(I + \mathbf{C}(e^{i\omega})\mathbf{P}(e^{i\omega}))^{-1}) \approx (\underline{\sigma}(\mathbf{C}(e^{i\omega})))^{-1}$ [23, Sec. 5.5], showing that $\sigma_n(\mathbf{J}_{par}(e^{i\omega}))$, and therefore $\bar{\sigma}(\mathbf{M}(e^{i\omega}))$, depends arbitrarily on the scaling of P and C.

For SAAILC, a serial ILC structure is proposed instead, see Fig. 1. The ILC input is added to the reference of the closed loop system such that $J_{ser} = PC(I + PC)^{-1}$ is the complementary sensitivity function. For frequencies where $\underline{\sigma}(\mathbf{P}(e^{i\omega})\mathbf{C}(e^{i\omega})) \gg 1$, $\underline{\sigma}(\mathbf{J}_{ser}(e^{i\omega})) \approx \bar{\sigma}(\mathbf{J}_{ser}(e^{i\omega})) \approx 1$, such that it is invariant under scaling of P and C. Therefore, $\bar{\sigma}(\mathbf{M}(e^{i\omega})) < 1$ for a large range of frequencies and directions, generally resulting in faster overall convergence for J_{ser} than for J_{par}. An additional advantage of serial ILC is that it allows C to contain integral action [24, Sec. 5].

VI. EXAMPLE

In this section, the stochastic approximation adjoint ILC algorithm is illustrated using two multivariable examples. SAAILC is compared to symmetric and non-symmetric deterministic MIMO approaches and to SPSA, and the differences between parallel and serial ILC are illustrated. Simulation models are used to ensure a fair and reproducible comparison.
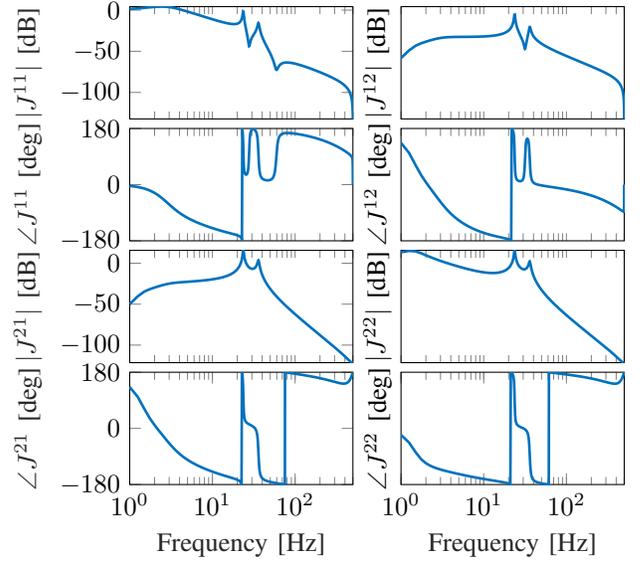

Fig. 2: Bode diagram of the transfer function $J = PC(I + PC)^{-1}$ for the gantry of the Arizona flatbed printer.

A. Setup

The SAAILC algorithm is illustrated using a 2×2 model of the gantry of an Arizona flatbed printer [3]. The Bode diagram of $J = J_{ser}$ using serial ILC is shown in Fig. 2. Note that the system is not symmetric. Disturbance r is given by $r = (I + PC)^{-1}y_d$, with reference y_d corresponding to a translation and a rotation of the gantry. In addition, SAAILC is illustrated on a random non-symmetric 21×21 MIMO system with 84 states, generated using the function `drss` in MATLAB. Disturbance r consists of a step in all directions.

B. Application of SAAILC

SAAILC is compared to the symmetric and non-symmetric deterministic MIMO algorithms and the SPSA approach from Section IV for the 2×2 printer in Fig. 3a. All algorithms use 1500 iterations with $W_e = I$, $W_f = 0$ and $\varepsilon_j = 0.5 \times \bar{\varepsilon}$.

In Fig. 3b, SAAILC (15000 iterations, $\varepsilon = 0.07\bar{\varepsilon}$) is compared to the symmetric (10000 iterations, $\varepsilon = 0.005\bar{\varepsilon}$) and non-symmetric (2500 iterations, $\varepsilon = 0.5\bar{\varepsilon}$) deterministic MIMO algorithms for a random 21×21 system. The smaller step size for SAAILC is necessary because of the increased variance. The smaller step size for symmetric deterministic MIMO ILC is chosen because for this system, assuming that J is symmetric is infeasbile and the cost diverges immediately. For both systems, it is shown that SAAILC achieves the same cost as the deterministic MIMO adjoint ILC algorithm, while reducing the required number of experiments significantly.

C. Comparison parallel and serial ILC

The convergence speed of MIMO deterministic adjoint ILC using a serial implementation is significantly higher than that for a parallel implementation, as is shown in Fig. 4. Therefore, it is recommended to use a serial ILC structure when using gradient descent ILC algorithms.
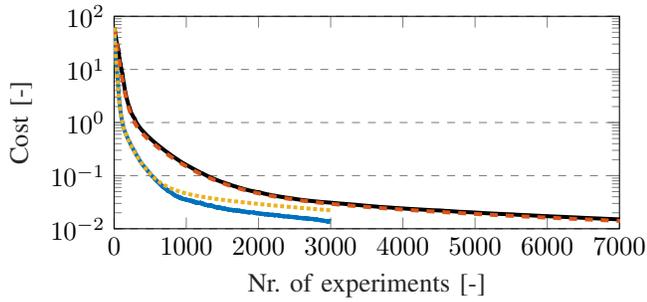
(a) Non-symmetric 2×2 flatbed printer.
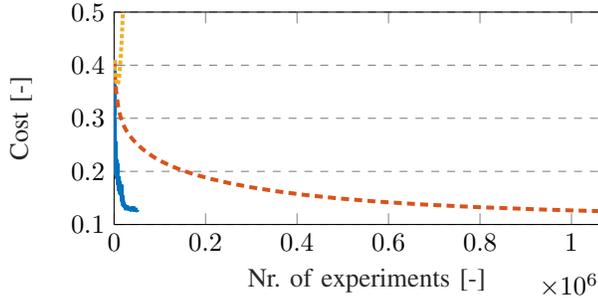
(b) Non-symmetric 21×21 system.

Fig. 3: The cost as a function of the number of experiments in adjoint ILC for non-symmetric 2×2 (a) and 21×21 (b) MIMO systems. SAAILC (—) requires far fewer experiments to reach the same cost as non-symmetric deterministic adjoint ILC (- -). Since the systems are non-symmetric, applying symmetric deterministic adjoint ILC (⋯) results in a higher (a) or diverging (b) cost. SPSA (—,a) does not reduce the number of experiments.

VII. CONCLUSIONS

A new method for model-free iterative learning control is introduced that is particularly suitable for massive MIMO systems. The stochastic approximation adjoint ILC algorithm uses a single dedicated experiment to estimate the gradient of a cost criterion, which is then used in a Robbins-Monro type stochastic gradient descent algorithm. Compared to the current deterministic MIMO approach, the SAAILC algorithm requires far fewer experiments to reach the same cost. Future developments involve the embedding in related iterative frameworks, including related stochastic approximation based schemes.

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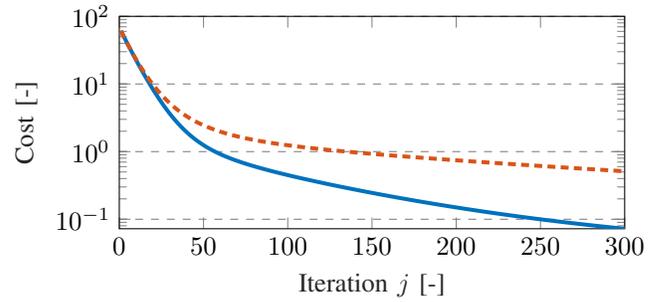
Fig. 4: The convergence of the cost using a gradient descent ILC algorithm with a serial ILC structure (—) is much faster than that with a parallel ILC structure (- -).

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